Linear Algebra

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Abstract

The lecture note of 2025 Fall Linear Algebra by professor 李明穗（Amy Lee）.

Contents

Chapter 0

Introduction

Lecture 1

0.1 Geometry

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- linear
- To study geometry with linearity
- In a different dimension:
  - In 2D: **lines**
  - In 3D: **planes**
  - In nD: **hyperplanes**

0.2 Abstract Algebra

Definition 0.2.1 (Linear Algebra)**.** Here is the definition of Linear algebra.

- Algebra is the study of basic "mathematical structure."
  e.g. **Group**, **Ring**, **Field**, ...etc.
- Linear Algebra studies one of the structures called **vector space**.

Note. Followed by logical deduction from the basic definition, we can derive some theorems.

0.3 Applied Science

- **Mathematic**: ODE, PDE.
- **Linear Programming**: developing during World War II
- **Imange Processing**, **Computer Vision**, **Computer Graphic**, etc.

Chapter 1

Matrices and Gaussian Elimination

1.1 Introduction

The central problem of Linear Algebra is the solution of Linear Equations. The most important and simplest case is when the # of unknowns equals to the # of equations.

Note. There are two ways to solve linear equations:

- The method of elimination **(Gaussian Elimination)**
- Determinants **(Crammer's Rule)**

1.1.1 Four aspects to follow

(1) The geometry of linear equations.

Note. $n = 2, n = 3 \quad \rightarrow \quad$ higher dimensional space.

(2) The interpretation of elimination is a factorization of the coefficient matrix.

Definition. Some notation to define:

Definition 1.1.1 (Scalar, Matrix, Vector).

$$Ax = b \qquad \begin{cases} \alpha, \beta, \gamma : & \text{scalar} \\ A, B, C : & \text{matrix} \\ a, b, c : & \text{vector} \end{cases}$$

Definition 1.1.2 (Lower/Upper triangular matrix).

$$A = LU \qquad \begin{cases} L : & \text{lower triangular matrix} \\ U : & \text{upper triangular matrix} \end{cases}$$

Definition 1.1.3 (Transpose/Inverse).

$$A^T / A^{-1} : \qquad \begin{cases} A^T : & \text{Transpose of matrix A} \\ A^{-1} : & \text{Inverse of matrix A} \end{cases}$$

(3) Irregular case and Singular case **(no unique solution)**:

> **Note.** no solution or infinitely many solutions

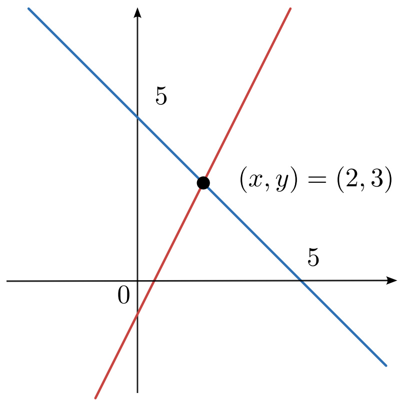
(4) **The # of operations to solve the system by elimination**

1.2 Geometry of Linear Equation

Example. Consider the linear equation below:

$$\begin{cases} 2x - y &= 1 \\ x + y &= 5 \end{cases}$$

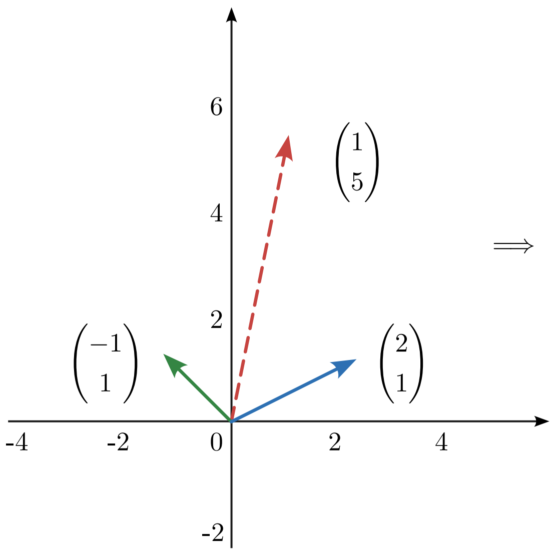
- **approach 1**: **row picture** $\to$ two lines in plane



$\Longrightarrow$ $(2, 3)$ is the intersection of two lines

Figure 1.1: Row Picture

- **approach 2**: **column picture**



$$\Longrightarrow \quad 2\begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Figure 1.2: Column Picture

Lemma 1.2.1 (Linear Combination).

$$x\begin{pmatrix}2\\1\end{pmatrix}+y\begin{pmatrix}-1\\1\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$$

To find the **Linear Combination** of $\begin{pmatrix}2\\1\end{pmatrix}$ and $\begin{pmatrix}-1\\1\end{pmatrix}$ to reach $\begin{pmatrix}1\\5\end{pmatrix}$

Note. A vector is a $n \times 1$ array with n real numbers, c_n is

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}$$

But in the text, we use

$$(c1, \cdots, c_n)$$

to represent.

Definition. Here are some operations on matrix:

Definition 1.2.1.

$$\alpha\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}_{n\times 1}=\begin{pmatrix}\alpha\cdot c_1\\\vdots\\\alpha\cdot c_n\end{pmatrix}_{n\times 1}, \qquad \alpha\in\mathbb{R}$$

Definition 1.2.2.

$$\begin{pmatrix}c_1\\\vdots\\c_n\end{pmatrix}+\begin{pmatrix}d_1\\\vdots\\d_n\end{pmatrix}=\begin{pmatrix}c_1+d_1\\\vdots\\c_n+d_n\end{pmatrix}_{n\times 1}$$

Definition 1.2.3.

$$y\in\mathbb{R}$$

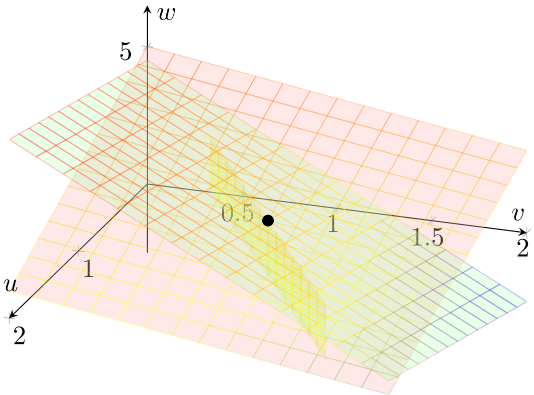
$$y\in\mathbb{R}^2 \implies y=\begin{pmatrix}y_1\\y_2\end{pmatrix}_{2\times 1} \qquad y_1, y_2\in\mathbb{R}$$

$$y\in\mathbb{R}^3 \implies y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix}_{3\times 1} \qquad y_1, y_2, y_3\in\mathbb{R}$$

Example. Consider the linear equation below:

$$\begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7u + 2w & = 9 \end{cases}$$

- Row picture



$$(u, v, w) = (1, 1, 2)$$

Lemma 1.2.2. in n-dimension, a line require $(n-1)$ equation.

Question. How to extend into n-dimensions?

Answer. Consider the following steps:

- Each equation represents a plane or hyperplane.
- The first equation produces a $(n-1)$-dimension plane in $\mathbb{R}^n$
- The second equation produces another $(n-1)$-dimension plane in $\mathbb{R}^n$
- Their intersection in smaller set of $(n-2)$-dimension
- $(n-3) \to (n-4) \to \cdots\cdots \to 3 \to 2 \to 1 \to$ point

Then we can find the final intersection. ⊛

- Column picture

$$u\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v\begin{pmatrix} 1 \\ -6 \\ 7 \end{pmatrix} + w\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \\ 9 \end{pmatrix} \quad \Longleftarrow \quad \begin{cases} 2u + v + w & = 5 \\ 4u - 6v & = -2 \\ -2u + 7u + 2w & = 9 \end{cases}$$

RHS is a linear combination of 3 column vectors.

Theorem 1.2.1. Solution to a linear equation:

$$\underbrace{(\textbf{intersection of to points})}_{\text{row pic.}} = \underbrace{(\textbf{coefficient of linear combination})}_{\text{column pic.}}$$

1.2.1 Singular Case

(1) Row Picture: In 3D case, they didn't intersect at a point.

- **Case 1**: two parallel

$$\begin{cases} 2u + v + w &= 5 \\ 4u + 2v + 2w &= 9 \end{cases}$$

- **Case 2**: three plane perpendicular ($\perp$)

$$\begin{cases} u + v + w &= 2 \ \cdots \ (1) \\ 2u + 3w &= 5 \ \cdots \ (2) \\ 3u + v + 4w &= 6 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1) + (2) = (3) \quad ; \quad \text{LHS} \Rightarrow (1) + (2) \neq (3)$$

- **Case 2**: three plane have a whole line in common.

$$\begin{cases} u + v + w &= 2 \ \cdots \ (1) \\ 2u + 3w &= 5 \ \cdots \ (2) \\ 3u + v + 4w &= 7 \ \cdots \ (3) \end{cases}$$

$$\text{RHS} \Rightarrow (1) + (2) = (3) \quad ; \quad \text{LHS} \Rightarrow (1) + (2) = (3)$$

- **Case 4**: three parallel

(2) Column Picture:

$$u \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + v \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + w \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = b$$

In the case above, three vectors are linear combination to each other, i.e. three vectors share the same plane.

Lemma 1.2.3 (Singular case)**.** If the three vectors are linear combination to each other (three vector share a common plane), it must be **singular case**.

- If $b = \begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix}$, which is on the plane $\Rightarrow$ too many solution to produce b.
- If $b = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix}$, which is not on the plane $\Rightarrow$ no solution.

1.2.2 Fundamental Linear Algebra Theorem (Geometry form)

Theorem 1.2.2 (Fundamental LA Theorem). Consider a linear system

$$Ax = b, \quad A \in \mathbb{R}^{m\times n},\ x \in \mathbb{R}^n,\ b \in \mathbb{R}^m.$$

If the n hyperplanes have no only one intersection or infinitely many points, then the n columns lie in the same plane. (consistency of *row picture* and *column picture*)

Notation. Logic notation:

- If ..., then : $\Rightarrow$
- If and only if : $\Leftrightarrow$

Lecture 2

1.3 An Example of Gaussian Elimination

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Example. Here is a linear equation.

$$\begin{cases} 2u + v + w &= 5 \\ 4u - 6v &= -2 \\ -2u + 7v + 2w &= 9 \end{cases}$$

$$\begin{pmatrix} \boxed{\mathbf{2}} & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & \boxed{\mathbf{-8}} & -2 & -12 \\ 0 & 8 & 3 & 14 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & \boxed{\mathbf{1}} & 2 \end{pmatrix} \quad \text{"pivot"}$$

Then we get $w = 2$, we can plug in the equation i.e.

$$\begin{cases} 2u + v + 1w = 5 \\ -8v - 2w = -12 \\ w = 2 \end{cases} \Longrightarrow \text{Forward Elimination}$$

Then we substitute into 2nd, 1st equation to get $v = 1$ and $u = 1 \Longrightarrow$ Backend Elimination

Note. By definition, pivots cannot be zero!

Question. Under what circumstances could the elimination process break down?

Answer. Here are some situations.

- Something **must** go wrong in the singular case.
- Something **might** go wrong in the nonsingular case.

A zero appears in a pivot position!
If in the process, there are nonzero pivots, then there's only one solution. ⊛

Example.

$$\begin{pmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{pmatrix}$$

(1) If $a_{11} = 0 \implies$ nonsingular

(2) If $a_{22} = 0 \implies$ nonsingular

(3) If $a_{33} = 1 \implies$ singular

Question. How many separate arithmetical operations does elimination require for n equations in n unknowns?

Answer. For a single operation.

$$\text{a single operation} = \text{each division \& each multiplication-subtraction}$$

⊛

- **FE**:

$$\underbrace{\begin{matrix} x & x & \cdots & x & = & x \\ \vdots & \vdots & & & & \vdots \\ x & x & \cdots & x & = & x \end{matrix}}_{n}$$

$$n(n-1) + (n-1)(n-2) + \cdots + (1^2 - 1) = \frac{n^3 - n}{3} \sim \frac{n^3}{3} \text{ steps}$$

- **RHS**:

$$(n-1) + (n-2) + \cdots + 1 = \frac{n(n-1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

- **BF**:

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \sim \frac{n^2}{2} \text{ steps}$$

1.4 Matrix Notation and Matrix Multiplication

$$\begin{cases} 2u + 4v + -2w &= 2 \\ 4u + 9v - 3w &= 8 \\ -2u - 3v + 7w &= 10 \end{cases} \implies u\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + v\begin{pmatrix} 4 \\ 9 \\ -3 \end{pmatrix} + w\begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}$$

We can rewrite it in the below form.

$$A = \underbrace{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}_{3\times 3}}_{\text{coefficient matrix}}, \quad x = \underbrace{\begin{pmatrix} u \\ v \\ w \end{pmatrix}_{3\times 1}}_{\text{unknowns}}, \quad b = \underbrace{\begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix}_{3\times 1}}_{\text{RHS}} \implies x = \underbrace{\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}_{3\times 1}}_{\text{solution}}$$

$$\boxed{Ax = b}$$

Definition 1.4.1. An $m \times n$ matrix, $A_{m\times n}$ over $\mathbb{R}$, is an array with m rows and n columns of real numbers, which can be written as

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \text{ where } a_{ij} \in \mathbb{R}, \begin{cases} i : \text{index of row} \\ j : \text{index of column} \end{cases}$$

- $\boxed{m \times n}$ is called the **dimensions (size)** of $A \implies$ dimension of a $()_{3\times 5}$ is 3×5
- $\boxed{a_{ij}}$ is called the **elements/entry/coefficient** of A
- **Addition**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{m\times n}$

$$A + B = (a_{ij} + b_{ij})_{m\times n}$$

- **Multiplication**: $A = (a_{ij})_{m\times n}, \quad B = (b_{ij})_{n\times r}$

$$AB = (c_{ij})_{m\times r}, \quad \text{where} \quad c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$$

- **Scalar Multiplication**:

$$\alpha A = (\alpha a_{ij})_{m\times n}$$

- $$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

  In particular, if

$$A_{1\times n} B_{n\times 1} = \mathbf{v} \cdot \mathbf{w} = ()_{1\times 1}.$$

  Then it's the **inner product** of vector $\mathbf{v}$ and vector $\mathbf{w}$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & 3 & -7 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2\cdot(-1) & 4\cdot(2) & -2\cdot(2) \\ 4\cdot(-1) & 9\cdot(2) & -3\cdot(2) \\ -2\cdot(-1) & 3\cdot(2) & -7\cdot(2) \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 22 \end{pmatrix}$$

$$(-1)\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} + 2\begin{pmatrix} 4 \\ 9 \\ 3 \end{pmatrix} + 2\begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$$

(1) by row: 3 inner product

(2) by column: a linear combination of 3 columns of A

Example (1A). Ax is a combination of columns of A

$$\begin{aligned} A_{m\times n}x_{n\times 1} &= \left(A_1|A_2|\cdots|A_n\right)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= x_1(A_1) + x_2(A_2) + \cdots + x_n(A_n) = \left(\sum_{j=1}^{n} a_{ij}\ x_j\right)_{m\times 1} \end{aligned}$$

1.4.1 The Matrix Form of One Elimination Step

Definition (1B). Matrix form

Definition 1.4.2. zero matrix:

$$O = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

Definition 1.4.3. identity matrix:

$$I = \begin{pmatrix} 1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1 \end{pmatrix} = I_n = I_{n\times n}; \quad \begin{cases} A_{m\times n}I_n = A_{m\times n} \\ A_{m\times n} = A_{m\times n}I_n \end{cases}$$

Definition 1.4.4. elementary matrix (elimination matrix):

$$E_{ij} = \begin{pmatrix} 1 & \cdots & 0 & \cdots & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ \vdots & & -\ell & \ddots & 0 \\ 0 & \cdots & \underset{\text{jth column}}{0} & \cdots & 1 \end{pmatrix} \text{ith row} \qquad \ell : \text{multiplier}$$

$$E_{ij} \cdot A = \begin{pmatrix} \cdots & -\ell & \cdots & 1 \\ & & & \end{pmatrix}\begin{pmatrix} \\ \\ \end{pmatrix} \begin{matrix} \longleftarrow \text{i-th} \implies (\text{i-th row}) + (-\ell)(\text{j-th column}) \\ \longleftarrow \text{j-th} \implies \text{create zero at } (i,j) \text{ position!} \end{matrix}$$

Example.

$$\underset{E_{21}}{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \underset{A}{\begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}} = \underset{EA}{\begin{pmatrix} 2 & 4 & -2 \\ 0_{21} & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix}}$$

Note. Here is two properties

1. $Ax = b \implies E_{ij}Ax = E_{ij}b$
2. $E_{ij}A \neq AE_{ij}$

1.4.2 Matrix Multiplication

(1) The (i, j)-th entry of AB is the inner product of the **i-th** of A and the **j-th** of B.

(2) Each column of AB is the product of a matrix A and **a column of B**

$$\begin{aligned} \implies \text{column } j \text{ of } AB &= A \text{ times } \textbf{j-th} \text{ of } B \\ &= \text{linear combination of } \textbf{columns of } A \\ &= b_{1j} A_{\underset{\text{any numbers}}{\boxed{\bullet}}\,1} + b_{2j}A_{\bullet 2} + \cdots + b_{nj}A_{\bullet n} \end{aligned}$$

Example.

$$\underset{A_{2\times3}}{\begin{pmatrix} 3 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}} \underset{B_{3\times3}}{\begin{pmatrix} 5 & 0 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 2 \end{pmatrix}} = \underset{C_{2\times3}}{\begin{pmatrix} 16 & 1 & 1 \\ 8 & 0 & -1 \end{pmatrix}}$$

$$\text{1st column of } AB = \begin{pmatrix} 16 \\ 8 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 2 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

(3) Each row of AB is a product of a row of A and a matrix B.

$$\implies \text{i-th row of } AB = \quad \text{of } A \text{ times } B.$$
$$= \text{linear combination of } \textbf{rows of } B$$
$$= a_{i1}B_{1\bullet} + a_{i2}B_{2\bullet} + \cdots + a_{in}B_{n\bullet}$$

Theorem 1.4.1. Let A, B and C be matrices (possibly rectangular). Assume that their dimension permit them to be added and multiplied in the following theorem

(1) The matrix multiplication is associative

$$(AB)C = A(BC)$$

(2) Matrix operations are distributive

$$A(B + C) = AB + AC$$

$$(A + B)C = AC + BC$$

(3) Matrix multiplication is noncommutative

$$AB \neq BA \quad \text{in general}$$

(4) Identity Matrix

$$A_{n\times n}I_n = I_nA_{n\times n} = A_{n\times n}$$

Example.

$$\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \underset{31}{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{1} & 0 & 1 \end{pmatrix}, \quad \underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-1} & 1 \end{pmatrix}$$

(1)

$$\underset{21}{E}\,\underset{31}{F} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \boxed{=} \underset{31}{F}\,\underset{21}{E} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

(2)

$$\underset{21}{E}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \boxed{\neq} \underset{32}{G}\,\underset{21}{E}$$

(3)

$$\underset{32}{G}\,\underset{31}{F}\,\underset{21}{E} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{pmatrix}}_{\text{"right order"}} \boxed{\neq} \underset{21}{E}\,\underset{31}{F}\,\underset{32}{G} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

Note. The product of lower triangular matrices is a lower triangular matrix.

Lecture 3

1.5 Triangular Factors and Row Exchanges

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$$\boxed{Ax = b}$$

$$\implies LUx = b \implies \begin{cases} Lc = b \\ Ux = c \end{cases}$$

Example.

$$Ax = \begin{pmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ 8 \\ 10 \end{pmatrix} = b$$

Remark. ℓ: multipliers

$$E_{ij}(\ell) : (\text{i-th row}) + (-\ell)(\text{j-th column})$$

$$\begin{pmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{pmatrix} \xrightarrow[R_3+(1)R_1]{R_2+(-2)R_1} \begin{pmatrix} 2 & 4 & -2 & 2 \\ 0_{21} & 1 & 1 & 4 \\ 0 & 1 & 5 & 12 \end{pmatrix} \xrightarrow{R_3+(-1)R_2} \begin{pmatrix} \boxed{2} & 4 & -2 & 2 \\ 0 & \boxed{1} & 1 & 4 \\ 0 & 0_{32} & \boxed{4} & 8 \end{pmatrix} \text{ pivot}$$

$$E_{21}(2) = E = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E_{31}(-1) = F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad E_{32}(1) = G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

i.e.

$$E_{21}E_{31}E_{32}Ax = \begin{pmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = Ux = c = \begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix} = E_{21}E_{31}E_{32}b$$

Question. How can we undo the steps of Gaussian Elimination?

$$E^{-1}F^{-1}G^{-1}GFEA = A = \underbrace{E^{-1}F^{-1}G^{-1}}\boxed{U} = LU \quad \text{i.e.} \quad A = \underset{\text{factors of } A}{LU}$$

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -(-2) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad F^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -(1) & 0 & 1 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -(-1) & 1 \end{pmatrix}$$

$$E^{-1}F^{-1}G^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \boxed{2} & 1 & 0 \\ \boxed{-1} & \boxed{1} & 1 \end{pmatrix} \implies \text{records everything that has been done so far}$$

1.5.1 Triangular Factorization

Theorem 1.5.1. If no exchanges are required, the original matrix A can be written as

$$A = LU$$

- The matrix L is lower triangular with 1's on the diagonal and the multipliers ℓ_{ij} (taken from elimiation) below the diagonal.
- The matrix U is the upper triangular matrix which appears after forward elimination and before back-substitution; its diagonal entries are the pivots.

Example.

$$\begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 6 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ 0 & -4 & 2 \\ 0 & 0 & 4 \end{pmatrix} \Rightarrow \text{提出2}$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 6 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

Question.

$$A = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 2 & 6 & 5 \\ -1 & 4 & -2 \\ 1 & 2 & 3 \end{pmatrix} \quad ; \quad A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

"triangular matrix" 有三條對角線

Answer. ⊛

1.5.2 One Linear System = Two Triangular Systems

$$Ax = b \implies Ux = c \;\&\; Lc = b \implies A = LU$$

Remark. The LU form is unsysttematic in one aspect. U has pivots along its diagonal where L always has 1's.
We can rewrite U as

$$U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & & u_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} = \begin{pmatrix} u_{11} & 0 & \cdots & 0 \\ 0 & u_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & u_{nn} \end{pmatrix} \begin{pmatrix} 1 & u_{12}/u_{11} & \cdots & u_{1n}/u_{11} \\ 0 & 1 & & u_{2n}/u_{22} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

Example.

$$A = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix} \begin{pmatrix} \boxed{3} & 4 \\ 0 & \boxed{2/3} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 1/3 & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2/3 \end{pmatrix}}_{D} \underbrace{\begin{pmatrix} 1 & 4/3 \\ 0 & 1 \end{pmatrix}}_{U}$$

Theorem 1.5.2. If

$$A = L_1 D_1 U_1 \text{ and } A = L_2 D_2 U_2$$

then

$$L_1 = L_2, D_1 = D_2, U_1 = U_2$$

i.e. if A has LDU decomposition, then it **is** unique.

1.5.3 Row Excahnge and Permutation Matrices

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \langle \text{Permutation matrix } P_{ij} \rangle$$

Note. Permutation matrix is also an elementary matrix.

Example. Here are some of the example:

1°
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} \quad \boxed{R_2 \leftrightarrow R_3}$$

2°
$$PA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{pmatrix} \quad \boxed{R_2 \leftrightarrow R_3}$$

3°
$$AP = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 4 \\ 0 & 3 & 0 \\ 0 & 5 & 6 \end{pmatrix} \quad \boxed{C_2 \leftrightarrow C_3}$$

Note. For the permutation matrix:

1° PA: Performing row exchange of A

2° AP: Performing column exchange of A

3° $PAx = Pb$; Should we permute the component of $x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ as well? **NONONONONO!!!**

Example.

$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ d & e & f \end{pmatrix} \qquad Ax = b$$

(1) if $d = 0$, the problem is incurable. The matrix is singular.

(2) if $d \neq 0$, $P_{13}A = \begin{pmatrix} d & e & f \\ 0 & 0 & c \\ 0 & a & b \end{pmatrix}$; if $a \neq 0$, $P_{23}P_{13}A = \begin{pmatrix} d & e & f \\ 0 & a & b \\ 0 & 0 & c \end{pmatrix}$

$$\begin{array}{c|ccc} & P_{23}P_{13} & \neq & P_{13}P_{23} \\ & 1 \quad 3 \quad 3 & & 1 \quad 1 \quad 2 \\ \text{row} & 2 \to 2 \to 1 & & 2 \to 3 \to 3 \\ & 3 \quad 1 \quad 2 & & 3 \quad 2 \quad 1 \end{array}$$

Theorem 1.5.3. We seperate into two cases:

- In the non singular case, there's a permutation matrix P that reorders the rows of A to avoid zeros in the pivot positions. In this case,
  1. (1) $Ax = b$ has a unique solution.
  2. (2) It is found by elimination with row exchange
  3. (3) With the rows reorders in advance, PA can be factored into LU $\langle PA = LU \rangle$
- In singular case, no reordering can produce a full set of pivots.

Example.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 3 \\ 2 & 5 & 8 \end{pmatrix} \xrightarrow[\cancel{\ell_{31}=2}\ \ell_{21}=2]{\cancel{\ell_{21}=1}\ \ell_{31}=1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 3 & 6 \end{pmatrix} \xrightarrow{P_{23}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \langle \text{This is WORNG} \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

To summarize: A good code for Gaussian Elimination keeps a record of L, U and P. They allow the solution $(Ax = b)$ from two triangular systems. If the system $Ax = b$ has a unique solution, they we say:

1° The system is nonsingular or

2° The matrix is nonsingular

Otherwise, it is singular.

1.6 Inverse and Transpose

Definition 1.6.1. An $n \times n$ matrix A is invertible if $\exists$ an $n \times n$ matrix $B \ni BA = I = AB$

Theorem 1.6.1. If A is invertible, then the matrix B satisfying $AB = BA = I$ is unique!

Proof. Suppose $\exists c \neq B \ni AC = CA = I$

$$B = BI = B(AC) = (BA)C = IC = C \text{ i.e } B = C$$

we call this matrix B, the $\boxed{\text{inverse of } A}$, and denoted as $\boxed{A^{-1}}$ ∎

Note. Not all $n \times n$ matrices have inverse.
e.g.

1°

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

2° if $Ax = \vec{0}$ has a $\underbrace{\text{nonzero solution}}^{x \neq 0}$, then A has no inverse!

$$x = A^{-1}(Ax) = A^{-1}\vec{0} = \vec{0} \quad (\rightarrow\leftarrow)$$

Note. The inverse of A^{-1} is A itself. i.e. $(A^{-1})^{-1} = A$.

Note. If $A = (a)_{1\times 1}$ and $a \neq 0$, then $A^{-1} = (\frac{1}{a})$. The inverse of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times 2}$ is

$$\frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{if } \det(A) \neq 0$$

Note.

$$A = \begin{pmatrix} d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & d_n \end{pmatrix} \quad d_i \neq 0,\ \forall i \quad \Longrightarrow A^{-1} = \begin{pmatrix} 1/d_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & 1/d_n \end{pmatrix}$$

Proposition 1.6.1. If A and B are invertible, then

- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A_1A_2\cdots A_n)^{-1} = A_n^{-1}\cdots A_2^{-1}A_1^{-1}$

1.6.1 The Calculation of A^{-1}: Gaussian-Jordan Method

$$A \cdot A^{-1} = I$$

$$A_{n\times n}B_{n\times n} = I_n$$

$$\begin{aligned}
&\Longrightarrow A_{n\times n}(B_1|B_2|\cdots|B_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\
&\Longrightarrow (AB_1|AB_2|\cdots|AB_n)_{n\times n} = (e_1|e_2|\cdots|e_n)_{n\times n} \\
&\Longrightarrow AB_1 = e_1;\ AB_2 = e_2;\ \cdots;\ AB_n = e_n \quad \longrightarrow \quad n \text{ linear systems:} Ax = b
\end{aligned}$$

Definition 1.6.2 (Gaussian-Jordan Method)**.** Instead of stopping at U and switching to back substitution, it continues by subtracting multipliers of a row from the rows above till it reaches a diagonal matrix. Then we divide each row by corresponding pivot.

$$(\underset{LU}{A}|I) \xrightarrow{\times L^{-1}} (U|L^{-1}) \xrightarrow{\times U^{-1}} (I|A^{-1})$$

$$\left(\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} \boxed{2} & -1 & 0 & 1 & 0 & 0 \\ 0 & \boxed{3/2} & -1 & 1/2 & 1 & 0 \\ 0 & 0 & \boxed{4/3} & 1/3 & 2/3 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & 1/4 \\ 0 & 1 & 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1 & 1/4 & 1/2 & 3/4 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}$$

1.6.2 Invertible = Nonsingular

Question. What kind of matrices are invertible?

Answer. Here are the example:

1° nonzero pivot Ch1 Ch4

2° nonzero determinants Ch4

3° independent columns (rows) Ch2

4° nonzero eigenvalues Ch5

which will in the whole course ⊛

Suppose a matrix A has full set of nonzero pivots. By definition, A is nonsingular and the n systems

$$Ax_1 = e_1,\ Ax_2 = e_2,\ \cdots\ , Ax_n = e_n$$

can be solved by elimination or Gaussian-Jordan Method.

Row exchanges maybe necessary, but the columns of A^{-1} are uniquely determined.

$$Ax = b \quad PAx = Pb$$
$$PAx_i = Pe_i$$

$$\{Pe_1, Pe_2, \cdots , Pe_n\} = \{e_1, e_2, \cdots , e_n\}$$

Note. Compute A^{-1}:

1° $A(x_1|\cdots|x_n) = I = (e_1|\cdots|e_n) \iff Ax_i = e_i,\ i = i\cdots n$

2° Gauss-Jordan Method: $(\ A \mid I\ ) \longrightarrow (\ I \mid A^{-1}\ )$

Question. We have found a matrix $A^{-1} \ni AA^{-1} = I$. But is $A^{-1}A = I$

Answer. We can do this by recall.

> **As previously seen.** Recall that every Gauss-Jordan step is a multiplication of matrices on the left. There are three types of elementary matrices:

1° $\boxed{E_{ij}(\ell)}$: to subtract a multiple ℓ of j row from i row.

2° $\boxed{P_{ij}}$: to exchange row i and j

3° $\boxed{D_i(d)}$: to multiply row i by d i.e. $D_i(d) = \begin{pmatrix} 1 & & & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & d & \vdots \\ \vdots & & \ddots & 0 \\ 0 & & & 1 \end{pmatrix}$ $\to$ ith row

$$\begin{pmatrix} d_1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix} = \begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$$

$$\implies DEEPEEA = I \implies A^{-1}A = I \therefore \text{we have a left inverse!}$$

These are the operation of A^{-1} ⊛

Theorem 1.6.2. For nonsingular and invertible:

- Every nonsingular matrix is invertible.
- Every invertible matrix is nonsingular.

Theorem 1.6.3. A square matrix is invertible $\iff$ it is nonsingular

Lecture 4

1.7 Transpose A^T

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Proposition 1.7.1. Here are the proposition of transpose

- $(A+B)^T = A^T + B^T$
- $(A^T)^T = A$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$

Proof. Here is the proof

1° $((A+B)^T)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = (A^T + B^T)_{ij}$

2° $((AB)^T)_{ij} = (AB)_{ji} = \boxed{\sum_{k=1}^{n} a_{jk} b_{ki}}$ $\quad (B^T A^T)_{ij} = \sum_{\ell=1}^{n} b^T_{i\ell} a^T_{\ell j} = \sum_{\ell=1}^{n} b_{\ell i} a_{j\ell} = \boxed{\sum_{\ell=1}^{n} a_{j\ell} b_{\ell i}}$

3°

■

Definition 1.7.1. A symmetric matrix is a matrix which equals its own transpose. i.e. $A = A^T$

Example.

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ YES } \begin{pmatrix} 5 & 4 \\ 1 & 5 \end{pmatrix} \text{ NO } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ YES}$$

Note. A symmetric matrix is not necessarily invertible. If it is invertible, then its inverse is symmetric.

Theorem 1.7.1. If A is symmetric and if A can be factored as LDU, then $A = LDU^T$

Proof. Here is the proof.

1° $A = A^T$, $A = LDU \Rightarrow A^T = (LDU)^T = U^T D^T L^T = A = LDU$

2° By theorem 1.5.2, the theorem is correct.

LDU is unique if they exist. ■

Chapter 2

Vector Spaces and Linear Equation

2.1 Vector Spaces and Subspace

To answer the basic questions about the $\underset{1^\circ}{\text{existence}}$ and $\underset{2^\circ}{\text{uniquness}}$ of the solution of $Ax = b$, we need the concept of vector space.

$$\text{Field} \implies \text{Vector Space} \implies \text{Solution of } Ax = b$$

Definition 2.1.1 (Field). Let F be a set with two operations "$+$" and "$\bullet$" i.e.

$$+ : F \times F \longrightarrow F$$
$$\bullet : F \times F \longrightarrow F$$

and $+, \bullet$ are well-defined functions. If the system $\boxed{(F, +, \bullet)}$ satisfies the following conditions, the F is called a $\boxed{\text{Field}}$.
For $a, b, c \in F$

(1) $(a + b) + c = a + (b + c)$

(2) $a + b = b + a$

(3) $\exists\, 0 \in F \ni a + 0 = 0 + a = a$ 單位元素 (1st operation)

(4) $\forall a \in F,\ \exists (-a) \in F \ni a + (-a) = 0$ 反元素 (1st operation)

(5) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

(6) $a \cdot b = b \cdot a$

(7) $\exists\, 1 \in F \ni a \cdot 1 = 1 \cdot a = a$ 單位元素 (2nd operation)

(8) $\forall a \neq 0 \in F,\ \exists\, a^{-1} \in F \ni a \cdot a^{-1} = a^{-1} \cdot a = 1$ 反元素 (2nd operation)

(9) $a \cdot (b + c) = ab + ac$ Distribution Law

Example.

$$\underset{\text{(real)}}{\mathbb{R}} \text{ (YES)} \quad \underset{\text{(rational)}}{\mathbb{Q}} \text{ (YES)} \quad \underset{\text{(integer)}}{\mathbb{Z}} \text{ (NO)} \quad \underset{\text{(complex)}}{\mathbb{C}} \text{ (YES)} \quad \mathbb{N} \text{ (NO)}$$

Definition 2.1.2 (vector space). Let V be a set and F be a field. V is a vector space over F if $\underset{1^\circ}{\text{addition}}$ and $\underset{2^\circ}{\text{multiplication by scalar}}$ are defined on V and they satisfy.

$$+ : V \times V \longrightarrow V$$
$$\bullet : F \times V \longrightarrow V$$

(A1) addition is associated

(A2) addition is commutative

(A3) $\exists$ zero vector $\in V \ni 0 + v = v + 0, \ \forall v \in V$

(A4) $\forall v \in V, \ \exists(-v) \in V \ni (-v) + v = 0$

(M1) $1 \cdot v = v, \ v \in V, \ 1 \in F$

(M2) $(\lambda\mu) \cdot v = \lambda(\mu v) \ v \in V, \ \lambda, \mu \in F$

(M3) $\lambda(v_1 + v_2) = \lambda v_1 + \lambda v_2 \ v_1, v_2 \in V, \ \lambda \in F$

(M4) $(\lambda + \mu)v = \lambda v + \mu v \ v \in V, \ \lambda, \mu \in F$

2.1.1 Algebraic Rules of Vector Algebra

Question. $n \in \mathbb{N}$, $\mathbb{R}^n/\mathbb{R}$ ($\mathbb{R}^n$ over $\mathbb{R}$) is a vector space?

Answer. **YES** ⊛

Example.

$$\mathbb{C}^n/\mathbb{C}, \ \mathbb{C}^n/\mathbb{R}, \ \mathbb{R}/\mathbb{R}$$

Question. $M_{2\times 2}(\mathbb{R})/\mathbb{R}$ is a vector space?

$$M_{2\times 2}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ a, b, c, d \in \mathbb{R} \right\}$$

Answer. **YES** ⊛

Question. V is a vector space?

$$V = \{\text{all } 3 \times 3 \text{ symmetric matrices over } \mathbb{R}\}$$

Answer. **YES** ⊛

Question. $\mathbb{R}^\infty/\mathbb{R}, \mathbb{R}^\infty = \{(a_1, a_2, \cdots\cdots) \mid a_i \in \mathbb{R}\}$

Answer. **YES** ⊛

Question. Let $V = \{f \mid f$ is a real-valued function defined on $[0,1]\}$ define $(rf)(x) = r \cdot f(x),\ r \in \mathbb{R}$

Answer. **YES**

$$(\text{zero vector}) = (\text{zero function})$$

i.e. $f(x) = 0, \forall x \in [0,1]$ ⊛

Question. $V = \{\text{all positive } \mathbb{R}\}$

$$\begin{cases} x+y & = xy \\ c \cdot x & = x^c \end{cases}, \quad \text{is } V \text{a v.s. over } \mathbb{R}$$

Answer. **YES**

1° (A1) $(x+y)+z = x+(y+z)$

2° (A2) $(x+y) = xy = yx = (y+x)$

3° (A3) zero vector: $x + 1 = x$

4° (A4) x + $\frac{1}{x}$ = zero vector = 1

5° (M3) $\lambda(x+y) = (x+y)^\lambda = (xy)^\lambda = x^\lambda y^\lambda = (\lambda x)(\lambda y) = \lambda x + \lambda y$

6° (M4) $(\lambda + \mu) \cdot x = x^{(\lambda+\mu)} = x^\lambda \cdot x^\mu = \lambda x \cdot \mu x = \lambda x + \mu x$

All conditions apply. ⊛

2.1.2 subspace

Definition 2.1.3 (subspace). A subspace W of a vector space $(V, +, \cdot)$ over F is a nonempty subset of $V \ni (W, +, \cdot)$ itself is a vector space over F. W is a subspace of V over F if and only if W is closed under addition and scalar multiplication.

Question. Does the zero vector belong to subspace?

Answer. **YES**
$W = \{\text{zero vector}\}$ is the smallest possible vector space. ⊛

Remark. If W_1 and W_2 are subspaces of V over F. Then $W_1 \cap W_2 \neq \emptyset$

Note. If W is a subspace of V/F, then we use notation $W \leq V$.

Question. $V = \mathbb{R}^2/\mathbb{R}$ (xy-plane), What are the subspace of V?

Answer. Here are all subspace of V

(i) origin (one point)

(ii) $\mathbb{R}^2/\mathbb{R} \leq V$

(iii) all lines through origin

(iv) ~~2nd quadant~~ (no zero)

There are much more example. ⊛

Question. $V = M_{n\times n}(\mathbb{R})/\mathbb{R}$

$$S = \{n \times n \text{symmetric matrix}\}$$
$$U = \{n \times n \text{upper triangular matrix}\}$$
$$L = \{n \times n \text{lower triangular matrix}\}$$

Answer. **YES, YES, YES** ⊛

Theorem 2.1.1 (). Let V be a vector space over F. A nonempty subset W of V is a subspace of V, if and only if for each pair $x, y \in W$ and $\alpha \in F$:

1° The zero vector $\in W$.

2° $\alpha x + y \in W$

2.1.3 Column Space of A

Example.

$$A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$$

The first concern is to find all attainable r.h.s. vector b. For example:

$$\begin{pmatrix}1 & 4\\ 2 & 5\\ 3 & 6\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \begin{pmatrix}b_1\\ b_2\\ b_3\end{pmatrix} = u\begin{pmatrix}1\\ 2\\ 3\end{pmatrix} + v\begin{pmatrix}4\\ 5\\ 6\end{pmatrix}$$

Theorem 2.1.2. The system is solvable if and only if the vector b can be expressed as a combination of columns of A

Note. The columns of $A_{m\times n}$ are vectors in $\mathbb{R}^m$, the rows of $A_{m\times n}$ are vectors in $\mathbb{R}^n$.

Example. Let $\mathcal{C}(A) = \{\text{all combinations of columns of } A\}$. Then, $\mathcal{C}(A)$ is a subspace of $\mathbb{R}^m/\mathbb{R}$.

Proof. If b and $b' \in \mathcal{C}(A)$, $\exists\, x, x' \ni Ax = b \;\&\; Ax' = b'$

$$\forall \alpha \in \mathbb{R}, \quad A(\alpha x + x') = A(\alpha x) + A(x') = A\alpha x + Ax' = \alpha b + b' \in \mathcal{C}(A)$$

$\implies \mathcal{C}(A) \leq \mathbb{R}^m/\mathbb{R}$ ∎

Definition 2.1.4. $\mathcal{C}(A)$ is called the column space of A. Thus if $b \in \mathcal{C}(A)$, then $Ax = b$ is solvable.

- $A_{m\times n} = 0 \longrightarrow \mathcal{C}(A) = 0_{m\times 1}$
- $A_{m\times n} = I_m \longrightarrow \mathcal{C}(A) = \mathbb{R}^m$

Lecture 5

2.1.4 Nullspace of A

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Definition 2.1.5. Let $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\}$, then $\mathcal{N}(A) \leq \mathbb{R}^n/\mathbb{R}$. Then $\mathcal{N}(A)$ is called the null space of A.

Proof. We proof it with the Theorem 2.1.1

- zero vector is in the $\mathcal{N}(A)$
- $x,\ x' \in \mathcal{N}(A) \;\Rightarrow\; Ax = 0,\ Ax' = 0$

$$A(x + x') = Ax + Ax' = 0 + 0 = 0 \;\Rightarrow\; x + x' \in \mathcal{N}(A)$$

$$A(\alpha x) = \alpha Ax = \alpha \cdot 0 = 0 \;\Rightarrow\; \alpha x \in \mathcal{N}(A), \forall \alpha \in \mathbb{R} \quad \therefore \mathcal{N}(A) \leq \mathbb{R}^n/\mathbb{R}$$

∎

Note. The system $Ax = 0$ is called a homogeneous equation. (齊次)

Remark. The solution set of $Ax = b$ is NOT a subspace of $\mathbb{R}^n/\mathbb{R}$

$$x, x' \longrightarrow Ax = b,\ \ Ax' = b$$

$$A(x + x') = Ax + Ax' = 2b \neq b$$

Example.

$$\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \mathcal{N}(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$$

Example.

$$\begin{pmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \mathcal{N}(A) = \left\{ \begin{pmatrix} t \\ t \\ -t \end{pmatrix}, t \in (-\infty, \infty) \right\}$$

$$\begin{aligned} \mathcal{C}(A) &= \{\text{all combinations of columns of } A\} \\ &= \text{ column space of } A \leq \mathbb{R}^m/\mathbb{R} \end{aligned}$$

$$\begin{aligned} \mathcal{N}(A) &= \{x \in \mathbb{R}^n \mid Ax = 0\} \\ &= \text{ null space of } A \leq \mathbb{R}^n/\mathbb{R} \end{aligned}$$

2.2 The Solution of m Equations in n Unknows

For $ax = b, \quad a, b, x \in \mathbb{R}$

(i) if $a \neq 0 \ \Rightarrow \ x = \dfrac{b}{a}$, unique

(ii) if $a = 0,\ b = 0 \ \Rightarrow$ infinitely many solutions.

(iii) if $a = 0,\ b = 0 \ \Rightarrow$ no x exists.

Now, consider $Ax = b$, if A is a square, then (i), (ii), (iii) may occur.

(i) A^{-1} exists $\longrightarrow \ x = A^{-1}b$, unique

(ii) A is singular (undetermined case)

(iii) inconsistent case.

With a rectangular matrix A, $x = A^{-1}b$ will never happen!

Definition. Here is the definition of two similar jargon.

Definition 2.2.1 (row echelon matrix). An $m \times n$ matrix R is called a row echelon matrix if

(i) the nonzero rows come first and the pivots are the first nonzero etries in those rows.

(ii) below each pivot is a column of zeros

(iii) each pivot lies to the right of the pivot in the row above.

e.g.

$$\begin{pmatrix} \circledast & \circledast & \circledast & \circledast & \circledast \\ 0 & \circledast & \circledast & \circledast & 0 \\ 0 & 0 & 0 & \circledast & \circledast \end{pmatrix}$$

Definition 2.2.2 (row-reduced echelon matrix). An $m \times n$ matrix R is called a row-reduced echelon matrix if

(i) the nonzero rows come first and the pivots are the first nonzero etries in those rows; pivots are normalized to be 1.

(ii) Above & Below each pivot is a column of zeros

(iii) each pivot lies to the right of the pivot in the row above.

e.g.

$$\begin{pmatrix} \boxed{1} & 0 & \circledast & 0 & \circledast \\ 0 & \boxed{1} & \circledast & 0 & \circledast \\ 0 & 0 & 0 & \boxed{1} & \circledast \end{pmatrix}$$

Theorem 2.2.1. To any $m \times n$ matrix A, there exists a permutation matrix P, a lower triangular matrix L with unit diagnal and an $m \times n$ echelon matrix $U \ni PA = LU$
OR
Every $m \times n$ matrix A is row equivalent to a row echelon matrix.

- Case 1. Homogeneous Case. $b_{m\times 1}=0$

$$Ax = 0$$

We call the component of x , which correspond to columns with pivots the basic variables; and these correspond to columns with pivots the free variables.

$$\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{cases} \text{basic variables: } u, w \\ \text{free variables: } v, y \end{cases}$$

The basic variables are then expressed in terms of free variables.

$$\begin{cases} 3w + y = 0 \\ u + 3v + 3w + 2y = 0 \end{cases} \quad \Longrightarrow \quad \begin{cases} w = -\frac{1}{3}y \\ u = -3v - y \end{cases}$$

$$x = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -3v - y \\ v \\ -\frac{1}{3}y \\ y \end{pmatrix} = v \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}$$

– $\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ is obtain from x by setting $\begin{cases} v = 1 \\ y = 0 \end{cases}$

– $\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{3} \\ 0 \end{pmatrix}$ is obtain from x by setting $\begin{cases} v = 0 \\ y = 1 \end{cases}$

Theorem 2.2.2. If a homogeneous system $A_{m\times n}x = 0$ has more $\overset{n}{\text{unknowns}}$ than $\overset{m}{\text{equations}}$ (m $<$ n), it has a nontrivial solution.

$$(A_{m\times n}) \longrightarrow (A_{m\times n})$$

at most m pivot, at most m basic variables, at least $(n-m)$ free variables.

Note. The nullspace is a subspace of the same dimension as the number of free variables.

- Case 2. Inhomogeneous Case: $b \neq 0$

$$Ax = b \;\rightarrow\; Ux = c \text{ where } c = L^{-1}b$$

$$\begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 - 2b_2 + 5b_1 \end{pmatrix} \longrightarrow b_3 - 2b_2 + 5b_1 = 0$$

We know that $Ax = b$ is solvable $\Rightarrow$ $b \in \mathcal{C}(A)$

– 1 & 3: basic variables

– $\mathcal{C}(A)$ = the set of combinations of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ & $\begin{pmatrix} 3 \\ 9 \\ 3 \end{pmatrix}$

, which is also $\left\{ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \middle| \; b_3 - 2b_2 + 5b_1 = 0 \right\} \perp \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$

Example.

$$b = \begin{pmatrix} 1 \\ 5 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \Longrightarrow \begin{cases} w = 1 - \frac{1}{3}y \\ u = -2 - 3v - y \end{cases}$$

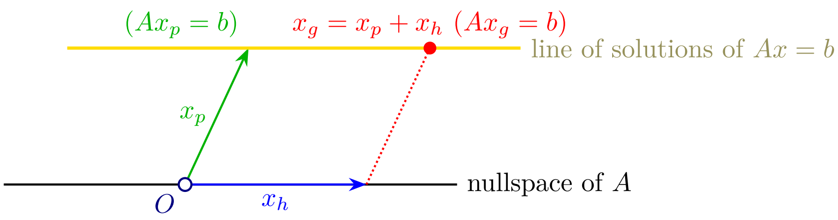
$$x = \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} -2 - 3v - y \\ v \\ 1 - \frac{1}{3}y \\ y \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\text{shift}} + \underbrace{v \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} -1 \\ 0 \\ -\frac{1}{3} \\ 1 \end{pmatrix}}_{\text{solution to } Ax=0 \text{ (nullspace)}}$$

Shift: particular solution to $Ax = b$ (set all free variables to be zero)

$$\boxed{x_{\text{general}} = x_{\text{particular}} + x_{\text{homogeneous}}; \;\; x_g = x_p = x_h}$$

Generally, the general solution is fills a two-dimensiona; surface (but NOT a subspace since it doesn't contain the zero vector (origin))

It is paralled to the Nullspace of A



2.2.1 Steps to oobtain the solution to $Ax = b$

(i) Reduce $Ax = b$ to $Ux = c$ to determine basic/free variables.

(ii) Set all free variables to zero to find particular solution, x_p

(iii) set RHS $= 0$. Give each free variables 1 others 0, in terms, find the hoomogeneous sloution, x_h

$$\implies \quad x_g = x_p + x_h$$

Definition 2.2.3 (rank). $A_{m\times n}$ if there are r pivots, there are r basic variables and $n-r$ free variables. The number of pivots, r, is called the rank of the matrix.

Theorem 2.2.3. Suppose elimination reduce $A_{m\times}x = b$ to $Ux = c$ and there are r pivots and the last $(m-r)$ rows of U are zero. Then there is a solution only if last $(m-r)$ elements of c are zeros.

- If $r = m$, there's always a solution. The general solution is the sum of particular solution and a homogeneous solution.
- If $r = n$, there are No free variables and the null space contains $x = 0$ only. The number r is called the rank of A.

Two extreme case: $A_{m\times n}x = b$

(1) If $r = n$ $\rightarrow$ No free variables $\rightarrow$ $\mathcal{N}(A) = \{x \in \mathbb{R}^n \mid Ax = 0\} = \{0\}$

(2) If $r = m$ $\rightarrow$ No zero rows in U $\rightarrow$ $\mathcal{C}(A) = \mathbb{R}^m$ $\Rightarrow$ $\exists$ solution for all b

2.3 Linear Independence, Basis and Dimension

In the elimination process, we refer to the number, r, of pivots as the rank of A. This definition is purely computational rather than mathematical. We shall give a formal definition later.
Now we shall disscuss the following four ideas:

(i) linear independence or dependence

(ii) **spanning** a subspace

(iii) **basis** for a subspace

(iv) **dimension** of a subspace

Definition 2.3.1. Let V be a vector space over F. A nonempty subset S of V is said to ve linearly dependent if there exist distinct vectors $v_1, v_2, \cdots, v_n$ in S and scalar $\alpha_1, \alpha_2, \cdots, \alpha_n$ in F, not all of which are zero $\ni$

$$\alpha_1 v_1 + \alpha_1 v_1 + \cdots + \alpha_n v_n = 0$$

A set which is not linearly dependent is called linearly independent. If $S = \{v_1, v_2, \cdots, v_n\}$ then we say that $v_1, v_2, \cdots, v_n$ are linearly dependent/independent.

Lecture 6

Remark (1). To show that $v_1, \cdots, v_n$ are linearly independent. We vaerify if 14 Oct. 13:20

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = 0 \text{ for some } c_i \in F$$

then c_i must be zero for all i.

Example. In $\mathbb{R}^2$, if v_1, v_2 are not colinear(共線) then they are linearly independent.

$$v_1 (\neq 0) \text{ and } v_2 (\neq 0) \text{ are linearly dependent } \iff v_1, v_2 \text{ are on the same } \textit{line}$$

Any 3 vectors in $\mathbb{R}^2$ are linearly dependent.

Remark (2). If $v_1 = v_2$, then the set $\{v_1, \cdots v_n\}$ is linearly dependent.

$$\alpha v_1 + (-\alpha) v_2 = 0$$

Remark (3). Any set which contain a linear dependent subset is linearly dependent.

Remark (4). Any subset of a linearly independent set is linearly independent.

Remark (5). Any set which contain 0 vector is linearly dependent.

Example.

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 3 & 2 & -3 & 0 \\ -4 & -4 & 2 & 1 \\ \underset{v_1}{-2} & \underset{v_2}{0} & \underset{v_3}{-4} & \underset{v_4}{0} \end{pmatrix}$$

The columns of A are linearly dependent.

$$c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0$$

$$(v_1 \ v_2 \ v_3 \ v_4) \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0 \quad \implies 4v_1 + (-3)v_2 + 2v_3 + 0v_4 = 0$$

Example.

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{pmatrix}$$

The columns of A are linearly independent

Note. We showed that the nullspace of A is $\{0\}$ only. That is exactly the same as saying the columns of A are linearly independent.

Example.

$$U = \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 2.3.1 (2F). The r nonzero rows of echelon matrix U are linearly independent, and so are r columns that contain pivots.

Example. In $\mathbb{R}^n$, $e_1, e_2, \cdots, e_n$ are linearly independent.

To summarize: To check any set of vectors $v_1, v_2, \cdots, v_n (\in \mathbb{R}^n)$ are linearly independent.
Let $A = (v_1 | v_2 | \cdots | v_n)_{m \times n}$, then solve $Ax_{n \times 1} = 0$.

1° if $\exists$ solution $x \neq 0$, then v_i's are linearly dependent.

2° if there are no free variables (i.e. $\text{rank}(A) = n$), nullspace $= \{0\}$ then v_i's are linearly independent.

3° if $\text{rank}(A) < n$, then v_i's are linearly dependent.

4° special case: if $v_i \in \mathbb{R}^{\gtrdot}$ and $n > m$, then v_i's are linearly dependent.

Proposition 2.3.2. A set of n vectors in $\mathbb{R}^m$ must be linearly dependent if $\boxed{n > m}$.

2.3.1 Spanning a Subspace

Definition 2.3.2 (2H). Let S be a subset vectors in V/F.
The subspace spanned by S is defined to be the intersection W of all subspaces of V which contain S.
When S is finite, $S = \{v_1, \cdots, v_n\}$, we call W the subspace spanned by $v_1, \cdots, v_n$ and denoted as $W = \langle v_1, \cdots, v_n \rangle$ or $W = \text{span}(S) = \langle S \rangle$.

Theorem 2.3.1. [The subspace spanned by a nonempty subset S] of a vector space V is [the set T of all linear combinations of vectors in S].

Proof. We need to show $W = T$.

Claim. $W = T$ if and only if $W \subseteq T$ and $T \subseteq W$.

- Let W be the subspace spanned by S, $S \subseteq W$ (S 不一定有包含 0 vector 所以不能用 $\leq$).
  So $\underset{T}{\text{every linear combination of vectors in } S}$ is in W. $\Longrightarrow T \subseteq W$.
  ($\because W$ is a subspace which is a vector space)
- on the other hand, T is a subspace containing S.
  ($\because x, y \in T,\ \alpha \in F \ \Rightarrow \ \alpha x + y \in T$)

So, $W \subseteq T$ by definition $\Rightarrow W = T$.
(Intersection of all subspaces containin S) ∎

Example. $\mathcal{C}(A) =$ space spanned by columns of A.

Example. $w_1 = (1, 0, 0),\ w_2 = (0, 1, 0),\ w_3 = (0, 0, 1)$, span a space $\mathbb{R}^3$.
$w_1 = (1, 0, 0),\ w_2 = (0, 1, 0),\ w_3 = (-3, 0, 0)$, span a plane $\mathbb{R}^2$.

Note. Spanning involves the columns space, independence involves the null space.

2.3.2 Basis

Definition 2.3.3 (2I). A basis for a vector space is a set of vectors that satisfies

(i) it is linearly independent AND

(ii) it span the vector space

If the basis of V is finite, then V is finite-dimensional (f-dim).

Remark (1). There's one and only one way to write every $v \in V$ as a linear combination of the basis elements.

Remark (2). In $\mathbb{R}^n$,

$$e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} \quad \begin{matrix} \uparrow \\ i^{th} \\ \downarrow \end{matrix}$$

then $\{e_1, \vdots, e_n\}$ is a basis for $\mathbb{R}^n$. The basis is called the standard basis.

$$\forall x = (x_1, \cdots, x_n) \in \mathbb{R}^n, \ x = \sum_{i=1}^{n} x_i e_i$$

The standard basis is not the only basis for $\mathbb{R}^n$. In fact, there are infinitely many bases for $\mathbb{R}^n$. For any nonsingular matrix $A_{n\times n}$, the columns of A are the basis for $\mathbb{R}^n$.

Example.

$$A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}_{3\times 4} \longrightarrow U = \begin{pmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{3\times 4}$$

The columns of U that contain pivots are a basis for $\mathcal{C}(U)$.

Note that $\mathcal{C}(U)$ is generate by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, which is a xy-plane within $\mathbb{R}^3$.

Remark. $\mathcal{C}(U)$ is NOT same as $\mathcal{C}(A)$.

Theorem 2.3.2 (2J). Any two bases for V contain the same number of vectors. This number is called the *dimension* of V.

Proof. Suppose $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ are bases for V, and suppose $m < n$.
For $j = 1, \ldots, n$,

$$w_j = a_{1j}v_1 + \cdots + a_{mj}v_m \quad \text{for some } a_{ij} \in F.$$

Let

$$w = [w_1, \ldots, w_n] = VA = [v_1, \ldots, v_m] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

The matrix A is $m \times n$ with $m < n$. By Theorem 2C, $\exists$ nontrivial C such that $AC = 0$.

$$VAC = WC = 0.$$

Hence the columns of W are linearly dependent. But the columns of W are basis elements, contradiction $\Rightarrow$ $m \not< n$.
Similarly, we can show that $n \not< m$, so we conclude $m = n$. ∎

Theorem 2.3.3 (2L)**.** Any linearly independent set in a finite-dimensional vector space V can be extended to a basis. Any spanning set of V can be reduced to a basis.

Proof. Let $v_1, \ldots, v_k$ be linearly independent over F. Then $\langle v_1, \ldots, v_k \rangle \leq V$.
If $\langle v_1, \ldots, v_k \rangle = V$, then $\langle v_1, \ldots, v_k \rangle$ is a basis of V. Otherwise, $\exists\, x \in V$ such that $x \notin \langle v_1, \ldots, v_k \rangle$.
Then $x, v_1, \ldots, v_k$ are linearly independent. If not, $\exists\, c \neq 0$, and $\exists\, \alpha_1, \ldots, \alpha_k$, not all zero, such that

$$cx + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0.$$

$$\Rightarrow \; x = c^{-1}\alpha_1 v_1 + c^{-1}\alpha_2 v_2 + \cdots + c^{-1}\alpha_k v_k.$$

$$\Rightarrow \; x \in \langle v_1, \ldots, v_k \rangle, \text{ contradiction.}$$

Then repeat the process, i.e. is $\langle x, v_1, \ldots, v_k \rangle = V$? Since V is finite-dimensional, the process will terminate after finite steps.
The 2nd half of the theorem can be proved similarly (exercise). ∎

2.4 The Four Fundamental Subspaces

Usually there are two ways to describe a subspace

(i) a set of vectors that span the space.
(e.g. the column space of $A_{m\times n}$, $\mathcal{C}(A)$)

(ii) a list of constraints that imposed on a subspace.
(e.g. the null space of $A_{m\times n}$, $\mathcal{N}(A) = \{x \mid Ax = 0\}$)

Here we will discuss four fundamental subspaces associated to $A_{m\times n}$

(1) the column space of A denoted by $\mathcal{C}(A)$

(2) the null space of A denoted by $\mathcal{N}(A)$

(3) the row space of A the columns spaces of A^T, denoted by $\mathcal{C}(A^T)$

(4) the left null space of A denoted by $\mathcal{N}(A^T)$, i.e. $\{y \mid A^T y = 0\}$

- If $A_{m\times n}$, then $\mathcal{C}(A), \mathcal{N}(A^T) \leq \mathbb{R}^m$ and $\mathcal{N}(A), \mathcal{C}(A^T) \leq \mathbb{R}^n$.

2.4.1 Row space $\mathcal{C}(A^T)$

The row space of A (the subspace spanned by the rows of A), $\mathcal{C}(A^T)$. For an echelon matrix, its r nonzero rows are independent and its row space is r-dimensional.

Proposition 2.4.1 (2M)**.** The row space of A has the same dimension r as the row space of echelon form U of A, and they have the same basis.

$$\mathcal{C}(A^T) = \mathcal{C}(U^T)$$

But in general, $\mathcal{C}(A) \neq \mathcal{C}(U)$.

Lecture 7

2.4.2 Nullspace $\mathcal{N}(A)$

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The nullspace of $A_{m\times n}$, $\{x \mid Ax = 0\} = \{x \mid Ux = 0\}$

$$\therefore \text{The nullspace of } A \text{ is the same as the nullspace of } U$$

Proposition 2.4.2 (2N). The nullspace $\mathcal{N}(A)$ is of dimension $n - r$
A basis of $\mathcal{N}(A)$ can be constructed by reducing to $Ux = 0$ which has $n - r$ free variables corresponding to the columns if U that do not contain pivots. Let each free variable 1, in turn, and others 0, and solve $Ux = 0$. The $n - r$ vectors produced in this manner will be a basis of $\mathcal{N}(A)$.

$$\dim(\mathcal{N}(A)) = n - r$$

The $\mathcal{N}(A)$ is also called the kernel of A, $\ker(A)$, and its dimension is called the nullity of A.

$$\ker(A) = \mathcal{N}(A)$$

2.4.3 Column space $\mathcal{C}(A)$

The $\mathcal{R}$ in $\mathcal{R}(A)$ stands for "**range**" which is consistent with the usual idea if range of f

Let $f(x) = A_{m\times n}x_{n\times 1}$, the

- the domain of f is $\mathbb{R}^n$
- the range of f is $\{b \in \mathbb{R}^m \mid Ax = b\} = \mathcal{C}(A) = \mathcal{R}(A)$
- the kernel of f is $\{x \in \mathbb{R}^n \mid Ax = f(x) = 0\} = \mathcal{N}(A) = \ker(A)$

If U is the echelon form of A, $\mathcal{C}(A) \neq \mathcal{C}(U)$, but they have the same dimension. For U, the columns with pivots form a basis of $\mathcal{C}(U)$. Then, the corresponding columns in A form a basis of $\mathcal{C}(A)$. Since the two systems $Ax = 0$, $Ux = 0$ are equivalent and have the same solutions. A nontrivial solution x means a linear combination of columns of U, hence the same linear combination of columns of A.

So, if the set of columns of U is independent, then so are the corresponding columns of A and vice versa.

To find a basis of $\mathcal{C}(A)$, we pick those columns of A, which correspond to the columns of U with pivots.

Proposition 2.4.3 (2O). The dimension of the column space = rank r, which also equals the dimension of the row space.

$$\therefore \#\text{ of independent columns} = \#\text{ of independent rows} = r$$

or more formally,

$$\text{rank}(A) = r = \text{row rank} = \text{column rank}$$

2.4.4 Left nullspace $\mathcal{N}(A^T)$

$$\underset{n\times m}{A^T}\underset{m\times 1}{y} = \underset{n\times 1}{0} = (\underset{1\times m}{y^T}\underset{m\times n}{A})^T$$

$$(\#\text{ of basic variables}) + (\#\text{ of free variables}) = (\#\text{ of variables}) = n$$

$$\boxed{\dim(\mathcal{C}(A)) + \dim(\mathcal{N}(A)) = \#\text{ of columns of } A}$$

For A^T, which has m columns, the column space of A^T is the row space of A which has dimension $\text{rank}(A)$. So,

$$\dim(\mathcal{N}(A^T)) = m - \text{rank}(A)$$

i.e.

$$\boxed{\dim(\mathcal{C}(A^T)) + \dim(\mathcal{N}(A^T)) = \#\text{ of columns of } A^T}$$

Proposition 2.4.4 (2P). The left nullspace $\mathcal{N}(A^T)$ is of dimension $m - r$
The left nullspace contain the coefficients that make the rows of A combined to a zero vector (linear dependent).

$$\text{To find } y \ni y^T A = 0$$

$$\text{Suppose that } PA = LU \longrightarrow \underset{m\times m}{\boxed{L^{-1}P}}\underset{m\times n}{A} = \underset{m\times n}{U}$$

The last $m - r$ rows of $L^{-1}P$ must be a basis for the left nullspace. ($\therefore$ the last $m - r$ rows of $L^{-1}P$ are independent and $\dim(\mathcal{N}(A^T))$ is $m - r \rightarrow$ it is a basis of $\mathcal{N}(A^T)$)

Theorem 2.4.1 (Fundamental Theorem of Linear Algebra). Let A be n arbitrary $m \times n$ matrix, then

$$\dim(\mathcal{C}(A)) = \dim(\mathcal{C}(A^T)) = \text{rank}(A)$$

$$\dim(\mathcal{N}(A)) = n - \text{rank}(A); \quad \dim(\mathcal{N}(A^T)) = m - \text{rank}(A)$$

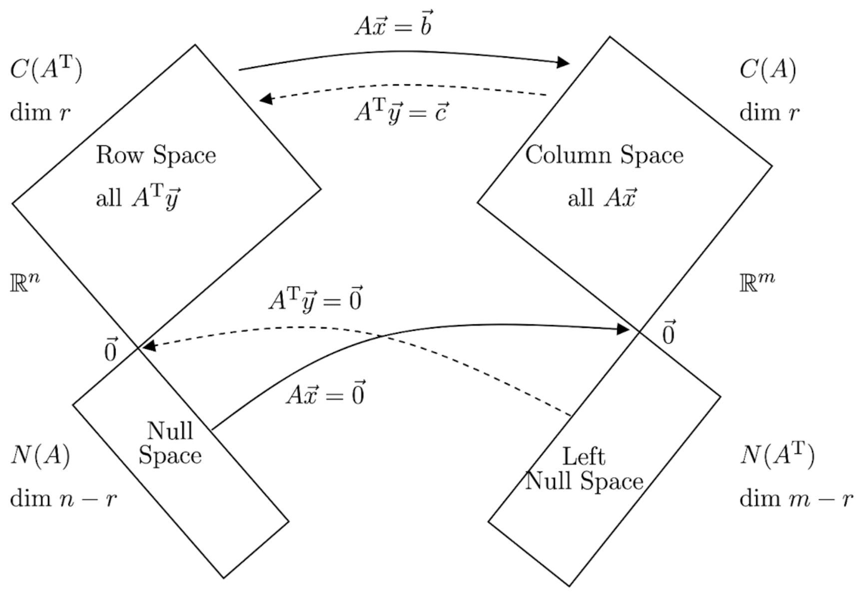


Figure 2.1: Fundamental Theorem of Linear Algebra

Example. Find out the basis for the four fundamental subspaces of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \\ 4 & 3 & 6 & 1 \end{pmatrix} \longrightarrow U = \begin{pmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad r = 2$$

1° $\mathcal{C}(A)$

$$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \right\} \quad \dim(\mathcal{C}(A)) = r = 2$$

2° $\mathcal{N}(A)$

$$Ax = 0 \longrightarrow Ux = 0 \longrightarrow U \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} x_1 + x_3 = 0 \\ x_2 + \frac{2}{3}x_3 + \frac{1}{3}x_4 = 0 \end{cases}$$

(a) $x_3 = 1,\ x_4 = 0 \longrightarrow \begin{pmatrix} -1 \\ -2/3 \\ 1 \\ 0 \end{pmatrix} = v_2$

(b) $x_3 = 0,\ x_4 = 1 \longrightarrow \begin{pmatrix} 0 \\ -1/3 \\ 0 \\ 1 \end{pmatrix} = v_2$

Hence, $\mathcal{B} = \mathcal{N}(A)$ is $\{v_1, v_2\}$ and

$$\dim(\mathcal{N}(A)) = n - r = 4 - 2 = 2$$

3° $\mathcal{C}(A^T)$

$$U = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 2/3 & 1/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} S_1 \\ S_2 \\ 0 \end{pmatrix} \longrightarrow \mathcal{B} = \{S_1^T, S_2^T\}, \quad \dim(\mathcal{C}(A^T)) = r = 2$$

4° $\mathcal{N}(A^T) \longrightarrow \mathcal{N}(B)$

$$B = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 3 \\ 1 & 4 & 6 \\ 0 & 1 & 1 \end{pmatrix} = A^T \longrightarrow \begin{pmatrix} \boxed{1} & 2 & 4 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow \begin{cases} y_1 + 2y_3 = 0 \\ y_2 + y_3 = 0 \end{cases}$$

$$z = 1 \longrightarrow \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \quad \therefore \mathcal{B} = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}, \quad \dim(\mathcal{N}(A^T)) = m - r = 3 - 2 = 1$$

Check orthogonality

Proposition 2.4.5 (2Q)**.** We can find the existence and uniqueness of solution of $Ax = b$.

- **Existence** of inverse:
  The system $Ax = b$ has at least one solution x for each b iff the columns span $\mathbb{R}^m$ $(r = m)$. In this case,
  $$\exists\, n \times m \text{ “right” inverse } C \ni AC = I$$
  This is possible only if $m \leq n$.

- **Uniqueness** of inverse:
  The system $Ax = b$ has at most one solution x for each b iff the columns are independent $(r = n)$. In this case,
  $$\exists\, n \times m \text{ “left” inverse } B \ni BA = I$$
  This is possible only if $m \geq n$.

Proof. We separetely prove the two parts.

- **Existence**:
  $$Ax = b \text{ has a solution for each } b \;\Leftrightarrow\; b \in \mathcal{C}(A), \forall b \in \mathbb{R}^m \;\Rightarrow\; \mathcal{C}(A) = \mathbb{R}^m$$
  Let $e_1, e_2, \cdots, e_m$ be the standard basis of $\mathbb{R}^m$.

  Then $\exists\, x_1, x_2, \cdots, x_m \ni Ax_i = e_i, \forall i = 1, 2, \cdots, m$

  Let $C = (x_1 \mid x_2 \mid \cdots \mid x_m)$, then $AC = A(x_1 \mid x_2 \mid \cdots \mid x_m) = (e_1 \mid e_2 \mid \cdots \mid e_m) = I_m$.

- **Uniqueness**:
  $$Ax = b \text{ has at most one solution for each } b \in \mathbb{R}^m$$
  $\Leftrightarrow$ $\forall b \in \mathbb{R}^m$, if b can be represented as linear combination of columns of A, then it is unique

Hence, proof is complete. ■

Example.

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix}_{2\times 3} \qquad m = 2, n = 3, r = 2 \quad \longrightarrow \quad \exists \text{ right inverse } C \ni AC = I$$

1°

$$AC = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \\ c_{31} & c_{32} \end{pmatrix} = I_2 \quad \Rightarrow \quad C \text{ is not unique}$$

2°

$$\begin{pmatrix} 1/4 & 0 \\ 0 & 1/5 \\ c_{31} & c_{32} \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{impossible since LHS is } 3 \times 2$$

3°

$$A_2 = \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix}_{3\times 2} \qquad m = 3, n = 2, r = 2 \quad \longrightarrow Ax = b \quad \begin{pmatrix} 4 & 0 \\ 0 & 5 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Note. The following statements about a square matrix $A_{n\times n}$ are equivalent:

(1) A is nonsingular (invertible)

(2) The columns of A span $\mathbb{R}^n$, so $Ax = b$ has only one solution $\forall b \in \mathbb{R}^n$

(3) The columns of A are independent, so $Ax = 0$ has only one trivial solution $x = 0$

(4) The rows of A span $\mathbb{R}^n$

(5) The rows of A are independent

(6) Elimination can be completed: $PA = LDU$ with all $d_i \neq 0$

(7) $\exists\ A^{-1} \ni AA^{-1} = A^{-1}A = I_n$

(8) Determinant of A $\det(A) \neq 0$

(9) Zero is NOT an eigenvalue of A

(10) A^TA is positive definite (正定)

2.5 Graph and Network

skip

2.6 Linear Transformation

We have seen that a matrix move subspaces around. For example, A maps $\mathcal{N}(A)$ to the zero vector and move all vectors into its column space $\mathcal{C}(A)$. Let A be an $n \times n$ matrix and $x \in \mathbb{R}^n$, so A transforms x into $Ax \in \mathcal{C}(A)$.

2.6.1 Notation of Linear Transformation

Example. Here are some examples of linear transformations:

1°

$$A = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \end{pmatrix} = c\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{(scaling by } c\text{)}$$

2°

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \quad \text{(rotation by } 90^\circ\text{)}$$

3°

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \quad \text{(reflection about } x_1 = x_2\text{)}$$

4°

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \quad \text{(projection onto } x_1\text{-axis)}$$

Lecture 8

Definition 2.6.1 (2T). Let V, W be vector spaces over a field $\mathbb{F}$. A linear transform from V to W is a function $T : V \to W$ such that preserves the operations on V and W, i.e. 28 Oct. 13:20

$$\begin{cases} T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), & \forall\ \mathbf{u}, \mathbf{v} \in V; \\ T(c\mathbf{u}) = cT(\mathbf{u}), & \forall\ \mathbf{u} \in V, c \in \mathbb{F}. \end{cases}$$

Example.

$$T : \mathbb{R}^3 \to \mathbb{R}^3$$

$$T : (x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$$

T is a linear transform.

Example.

$$A = \frac{d}{dt} : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n-1}(\mathbb{R})$$

$$p(t) \in \mathbb{P}_n(\mathbb{R}),\ p(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

See the attributes below:

$$AP = \frac{d}{dt}(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n) = a_1 + 2a_2 t + \cdots + n a_n t^{n-1}$$

The nullspace of A is all constant polynomials.

$$\mathcal{C}(AP) = \mathbb{P}_{n-1}(\mathbb{R})$$

the basis is $\{1, t, t^2, \ldots, t^{n-1}\}$ and $\operatorname{rank}(\mathcal{C}(A)) = n$.

$$\operatorname{nullity}(A) + \operatorname{rank}(A) = 1 + n = \dim(\mathbb{P}_n(\mathbb{R})).$$

Example.

$$A = \int_0^t : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n+1}(\mathbb{R})$$

See the attributes below:

$$AP = \int_0^t (a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)\ dt = a_0 t + \frac{a_1 t^2}{2} + \frac{a_2 t^3}{3} + \cdots + \frac{a_n t^{n+1}}{n+1} + C$$

The nullspace of A is all constant polynomials.

$$\mathcal{N}(AP) = \{0\}$$

The range of A

$$\mathcal{C}(AP) = \mathbb{P}_{n+1}(\mathbb{R}) - \{\text{constant}\}/\{0\}$$

Example.

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$T : (x_1, x_2, x_3) \mapsto 2x_1 + 3x_2 - x_3,\ \ x_i \in \mathbb{R}$$

T is a linear transform.

Example.

$$T : \mathbb{R}^3 \to \mathbb{R}$$

$$T : (x_1, x_2, x_3) \mapsto 2x_1^2 + 3x_2 - x_3,\ x_i \in \mathbb{R}$$

T is NOT a linear transform.

$$\because\ T(x+y) \neq T(x) + T(y)$$

Theorem 2.6.1. Let $T : V \to W$ be a linear transform, where V, W are vector spaces over a field $\mathbb{F}$.

(i) If M is a subspace of V, then

$$T(M) = \{x \in W \mid \exists\ \mathbf{m} \in M,\ \text{such that } T(\mathbf{m}) = x\}$$

is a subspace of W.

(ii) If N is a subspace of W, then

$$T^{-1}(N) = \{\mathbf{v} \in V \mid T(\mathbf{v}) \in N\}$$

is a subspace of V.

Proof. Here is the proof:

(i) Let $M \leq V$, $y_1, y_2 \in T(M) \subseteq W$, and $\alpha \in \mathbb{F}$.

$$y_1, y_2 \in T(M) \Rightarrow \exists x_1, x_2 \in M \text{ s.t. } T(x_1) = y_1,\ T(x_2) = y_2$$

Then

$$T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2)$$

since T is a linear transformation.
Also

$$\alpha x_1 + x_2 \in M$$

since M is a subspace of V.
Therefore

$$\alpha y_1 + y_2 = \alpha T(x_1) + T(x_2) = T(\alpha x_1 + x_2) \in T(M)$$

so $T(M)$ is a subspace of W.

(ii) Let $x_1, x_2 \in T^{-1}(N)$ and $\alpha \in \mathbb{F}$.

$$T(\alpha x_1 + x_2) = \alpha T(x_1) + T(x_2) \in N$$

since $N \leq W$ and $T(x_1), T(x_2) \in N$.
Therefore

$$\alpha x_1 + x_2 \in T^{-1}(N)$$

and $T^{-1}(N)$ is a subspace of V.

■

Definition 2.6.2. $T : V \to W$ over a field $\mathbb{F}$ is a linear transform. Then $T^{-1}(\mathbf{O}_W)$ is called the nullspace (kernel) of T, where $\mathbf{O}_W$ is the zero vector in W. $T(V)$ is called the range (image) of T.

$$\dim(T^{-1}(\mathbf{O}_W)) = \text{nullity}(T)$$

$$\dim(T(V)) = \text{rank}(T)$$

2.6.2 Matrix Representation of Linear Transformations

Question. What is the transformation taken $A : \mathbb{R}^2 \to \mathbb{R}^3$

$$x_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_{\in\mathbb{R}^2} \to \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}_{\in\mathbb{R}^3}, \quad x_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{\in\mathbb{R}^2} \to \begin{pmatrix} 4 \\ 6 \\ 8 \end{pmatrix}_{\in\mathbb{R}^3}$$

Answer.

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 6 \\ 4 & 8 \end{pmatrix}_{3\times 2} = (T)_{\{e_1,e_2\}}^{\{e_1,e_2,e_3\}}$$

⊛

Example.

$$T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_2(\mathbb{R}), \quad \text{i.e. } T(f) = \frac{d}{dt}(f)$$

The ordered basis of two vector spaces are

$$\begin{cases} \mathcal{B}_1 = \mathcal{B}(\mathbb{P}_3(\mathbb{R})) : & \{1, t, t^2, t^3\} \\ \mathcal{B}_2 = \mathcal{B}(\mathbb{P}_2(\mathbb{R})) : & \{1, t, t^2\} \end{cases}$$

Then we have

$$\left(T\right)_{\mathcal{B}_2}^{\mathcal{B}_1} = \begin{matrix} 1 \\ t \\ t^2 \end{matrix} \begin{pmatrix} \overset{1}{0} & \overset{t}{1} & \overset{t^2}{0} & \overset{t^3}{0} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}_{3\times 4} \qquad \text{e.g. } \left(T\right) \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}_{3\times 1}$$

Example.

$$T : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_3(\mathbb{R}), \quad \text{i.e. } T(f) = \frac{d}{dt}(f)$$

We have to handle the t^3 term, which means

$$\left(T\right)_{\mathcal{B}_1}^{\mathcal{B}_1} = \left(T\right)_{\mathcal{B}_1} = \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \end{matrix} \begin{pmatrix} \overset{1}{0} & \overset{t}{1} & \overset{t^2}{0} & \overset{t^3}{0} \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{4\times 4}, \qquad \text{this is called a } \langle\text{differentiation matrix}\rangle$$

Example.

$$\int_0^t : \mathbb{P}_3(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$$

The ordered basis of two vector spaces are

$$\begin{cases} \mathcal{B}_1 = \mathcal{B}(\mathbb{P}_3(\mathbb{R})) : \quad \{1, t, t^2, t^3\}, \quad \dim(\mathbb{P}_3(\mathbb{R})) = 4 \\ \mathcal{B}_2 = \mathcal{B}(\mathbb{P}_4(\mathbb{R})) : \quad \{1, t, t^2, t^3, t^4\}, \quad \dim(\mathbb{P}_4(\mathbb{R})) = 5 \end{cases}$$

hence we have

$$\left(T\right)_{\mathcal{B}_2}^{\mathcal{B}_1} = \begin{matrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{matrix} \begin{pmatrix} \overset{1}{0} & \overset{t}{1} & \overset{t^2}{0} & \overset{t^3}{0} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}_{5\times 4}$$

which is called an ⟨integration matrix⟩

and we also have

$$\mathcal{C}(T) = \operatorname{span}\{t, t^2, t^3, t^4\}, \quad \operatorname{rank}(T) = 4$$

$$\mathcal{N}(T) = \{0\}, \quad \operatorname{nullity}(T) = 0$$

Example.

$$\mathbb{P}_2(\mathbb{R}) \xrightarrow{\int_t} \mathbb{P}_3(\mathbb{R}) \xrightarrow{\frac{d}{dt}} \mathbb{P}_2(\mathbb{R})$$

$$\left(\frac{d}{dt} \int_0^t\right) = \left(\frac{d}{dt}\right)_{3\times 4} \left(\int_0^t\right)_{4\times 3} = \left(I\right)_{3\times 3} \quad \text{Diff is the left inverse of Int}$$

2.6.3 Rotation Q, Projection P, Reflection R

We introduce three important linear transformations in $\mathbb{R}^2$:

$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

1° Rotation: Q rotates vectors by an angle θ.

$$\left(Q\right) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}_{2\times 2}$$

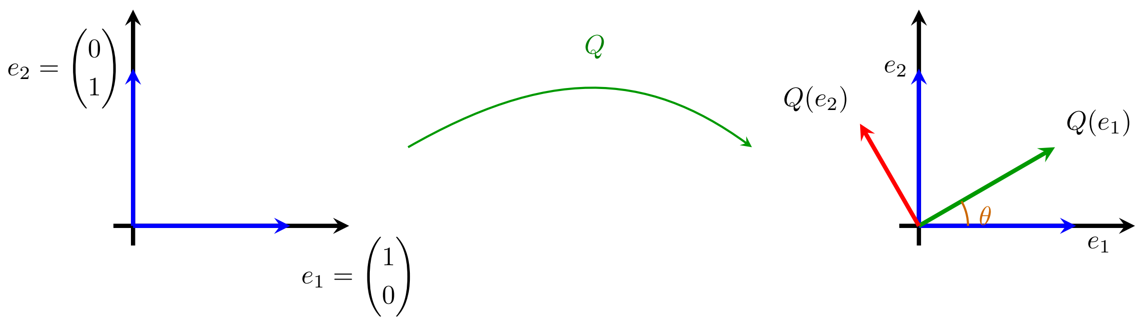
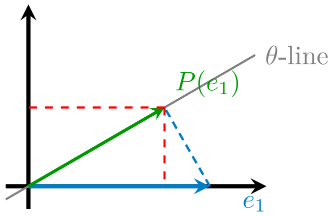


Figure 2.2: Rotation in $\mathbb{R}^2$

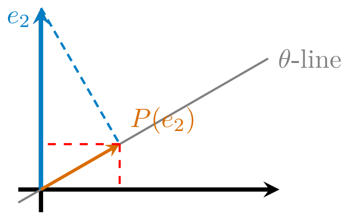
$$Q\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix},\quad Q\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}$$

- $Q_{-\theta}\cdot Q_\theta=\mathbf{1}_{\mathbb{R}^2}$
- $Q_\theta\cdot Q_\theta=Q_{2\theta}$
- $Q_\theta\cdot Q_\phi=Q_{\theta+\phi}$

2° Projection: P projects vectors onto the θ-line.



$$P(e_1)=\begin{pmatrix}\cos^2\theta\\\cos\theta\sin\theta\end{pmatrix}$$



$$P(e_2)=\begin{pmatrix}\sin\theta\cos\theta\\\sin^2\theta\end{pmatrix}$$

Figure 2.3: Projection onto a line at angle θ

$$P=\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$$

Here are some properties of projection:

- $P^2=P$
- Symmetric: $P^T=P$
- P^{-1} does not exist.

1°

$$\begin{aligned}P\left(\alpha\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\right)&=\alpha P\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\\&=\alpha\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}=\alpha\begin{pmatrix}\cos^3\theta+\cos\theta\sin^2\theta\\\sin\theta\cos^2\theta+\sin^3\theta\end{pmatrix}=\alpha\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}\end{aligned}$$

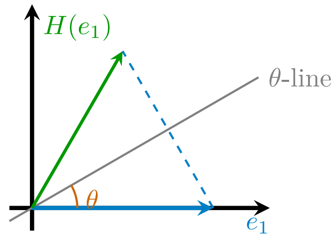
2°

$$\begin{aligned}P\left(\alpha\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\right)&=\alpha P\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\\&=\alpha\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\\sin\theta\cos\theta & \sin^2\theta\end{pmatrix}\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}=\alpha\begin{pmatrix}-\sin\theta\cos^2\theta+\cos\theta\sin\theta\\-\sin^2\theta\cos\theta+\sin^3\theta\end{pmatrix}=\alpha\begin{pmatrix}0\\0\end{pmatrix}\end{aligned}$$

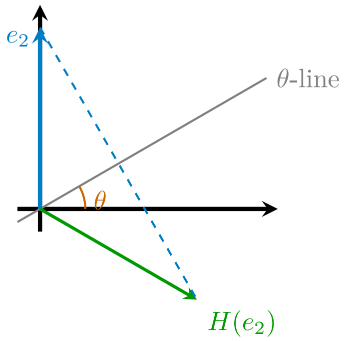
Thus,

$$\boxed{\alpha\begin{pmatrix}-\sin\theta\\\cos\theta\end{pmatrix}\quad\text{is in the nullspace of } P.}$$

3° Reflection: R reflects vectors across the θ-line.



$$H(e_1) = \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos^2\theta - 1 \\ 2\sin\theta\cos\theta \end{pmatrix}$$



$$H(e_2) = \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + \sin\theta \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sin\theta\cos\theta \\ 2\sin^2\theta - 1 \end{pmatrix}$$

Figure 2.4: Reflection across a line at angle θ

$$H = \begin{pmatrix} 2\cos^2\theta - 1 & 2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & 2\sin^2\theta - 1 \end{pmatrix}$$

Here are some properties of reflection:

- $H^2 = I$
- $H^{-1} = H$
- $H = 2P - I$ $(Hx + x = 2Px)$

Note. If first basis vector is on the θ-line, and the second basis vector is perpendicular to the θ-line, then

$$P^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad H^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2P^* - I, \quad Q^* = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Chapter 3

Orthogonality

3.1 Perpendicular Vectors and Orthogonal Subspaces

There are three important concepts in this section:

(i) The length of vector

(ii) The test for perpendicularity

(iii) How to create perpendicular vectors form linearly independent vectors

Now we start to discuss:

(i) **The length of vector**:
The length (or norm) of a vector, in $\mathbb{R}^n$, that satisfies the Pythagorean theorem is defined as:

> **Definition 3.1.1.** Let $\mathbf{x} \in \mathbb{R}^n$ be
>
> $$\mathbf{x} = (x_1, x_2, \dots, x_n) = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n \in \mathbb{R}^n$$
>
> then
>
> $$\|\mathbf{x}\|^2 = \sum_{i=1}^{n} x_i^2 = \mathbf{x}^T \mathbf{x}$$

(ii) **The test for perpendicularity**:

> **Definition 3.1.2.** Given $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then if $\mathbf{x} \perp \mathbf{y}$, then by Pythagorean theorem, we have
>
> $$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = \|\mathbf{x} - \mathbf{y}\|^2$$
>
> Then we can deduce that
>
> $$x_1^2 + x_2^2 + \cdots + x_n^2 + y_1^2 + y_2^2 + \cdots + y_n^2 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2$$
>
> then we have
>
> $$\mathbf{x}^T \mathbf{y} = 0$$

Definition 3.1.3 (Inner Product)**.** Let V be a vector space over a field $\mathbb{F}$ $(\mathbb{R}, \mathbb{C})$. An inner product on V is a function that assigns to every ordered pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in V, a scalar in $\mathbb{F}$, denoted as

$$\langle \mathbf{x}, \mathbf{y} \rangle$$

$\forall\ \mathbf{x}, \mathbf{y}, \mathbf{z} \in V,\ c \in \mathbb{F}$, we have

(a) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$

(b) $\langle c\ \mathbf{x}, \mathbf{y} \rangle = c\ \langle \mathbf{x}, \mathbf{y} \rangle$

(c) $\langle \mathbf{y}, \mathbf{x} \rangle = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$ (where $\overline{a + bi} = a - bi$ complex conjugate)

(d) $\langle \mathbf{x}, \mathbf{x} \rangle > 0$, if $\mathbf{x} \neq \mathbf{0}$

Note (1)**.** If $\mathbb{F} = \mathbb{R}$, (c) will reduce to $\langle \mathbf{y}, \mathbf{x} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$.

Note (2)**.** Inner product is linear in the first component.

Definition 3.1.4 (Standard Inner Product)**.** Let $V = \mathbb{R}^n/\mathbb{R}$, defined

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{y} = \sum_{i=1}^{n} x_i y_i$$

This is called the standard inner product on $\mathbb{R}^n$.

Proposition 3.1.1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then

- Let $\langle \mathbf{x}, \mathbf{y} \rangle = x^T y$ be standard inner product.
- Let $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ if and only if $\mathbf{x} \perp \mathbf{y}$.

Example. If $\langle , \rangle$ is any inner product on V, and $r > 0$, we define

$$\langle \mathbf{x}, \mathbf{y} \rangle' = r \langle \mathbf{x}, \mathbf{y} \rangle$$

1° $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle' = r\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = r\langle \mathbf{x}, \mathbf{z} \rangle + r\langle \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle' + \langle \mathbf{y}, \mathbf{z} \rangle'$

2° $\langle c\mathbf{x}, \mathbf{y} \rangle' = r\langle c\mathbf{x}, \mathbf{y} \rangle = c \cdot r\langle \mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle'$

Example. Let $V = \{f \mid f :$ real-valued continuous functions on $[0, 1]\} = \mathcal{C}([0, 1])$. For $f, g \in V$, define

$$\langle f, g \rangle = \int_0^1 f(t)g(t)dt$$

Example. Let $V = \mathbb{C}^n$, $\mathbb{C}^n$ is a vector space over $\mathbb{C}$. For $\mathbf{x}, \mathbf{y} \in V$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \overline{\mathbf{y}} = \sum_{i=1}^{n} x_i \overline{y_i}$$

$$\langle \mathbf{y}, \mathbf{x} \rangle = \mathbf{y}^\mathsf{T} \overline{\mathbf{x}} = \overline{\mathbf{x}^\mathsf{T} \overline{\mathbf{y}}} = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}.$$

Example. Let $V = \mathbb{C}$, $\mathbb{C}$ is a vector space over $\mathbb{C}$. If $\mathbf{x}, \mathbf{y} \in \mathbb{C}$, $x = a + bi$, $y = c + di$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = (a + bi)(c - di)$$

1° $\langle \mathbf{y}, \mathbf{x} \rangle = (c + di)(a - bi) = \overline{(a + bi)(c - di)} = \overline{\langle \mathbf{x}, \mathbf{y} \rangle}$

2° $\langle \mathbf{x}, \mathbf{x} \rangle = (a + bi)(a - bi) = a^2 + b^2 > 0$ if $\mathbf{x} \neq 0$

Lecture 9

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Definition 3.1.5 (inner product space)**.** An inner product space is a real or complex vector space (i.e. a vector space over the field $\mathbb{R}$ or $\mathbb{C}$) together with a specified inner product on that space.

Definition 3.1.6 (orthogonal)**.** In an inner product space V, $\mathbf{x}$ is orthogonal to $\mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. A set S of vectors in V is called orthogonal set if all pairs of distinct vectors in S are orthogonal. An orthonormal set is an orthogonal set of unit vectors.

$$\langle \mathbf{v}, \mathbf{v} \rangle = \|\mathbf{v}\|^2 = 1, \quad \forall \mathbf{v} \in S$$

Proposition 3.1.2. An orthogonal set of nonzero vectors is linearly independent.

Proof. Let $v_1, \cdots, v_n$ be nonzero distinct vectors in S, and $c_1, \cdots, c_n \in \mathbb{F}$

$$c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \sum_{i=1}^{n} c_i v_i = y$$

$$\langle y, v_j \rangle = \left\langle \sum_{i=1}^{n} c_i v_i, v_j \right\rangle = \sum_{i=1}^{n} c_i \langle v_i, v_j \rangle = c_j \langle v_j, v_j \rangle = c_j \|v_j\|^2$$

Then we have $y = 0 \iff c_j = 0, \ \forall j$

$$\therefore \{v_1, v_2, \cdots, v_n\} \text{ is linearly independent.}$$

■

Example. $\{e_1, e_2, \cdots, e_n\}$ is an orthonormal set (basis) for $\mathbb{R}^n$

In $\mathbb{R}^2$,

1° $\{e_1, e_2\}$

2° $\left\{ \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

3° $\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \right\}$

3.1.1 Orthogonal Subspaces

Definition 3.1.7 (3B). Let W_1 and W_2 be subspaces of an inner product space V. We say that W_1 is orthogonal to W_2 ($W_1 \perp W_2$) if

$$\langle \mathbf{w}_1, \mathbf{w}_2 \rangle = 0, \quad \forall \mathbf{w}_1 \in W_1, \forall \mathbf{w}_2 \in W_2$$

Note. In $\mathbb{R}^3$, the xy-plane is NOT orthogonal to the yz-plane. Because vectors along the y-axis are in both planes, and their inner product is not zero.

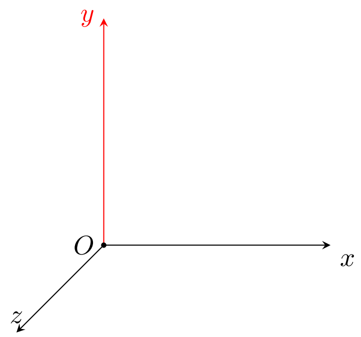


Figure 3.1: The xy-plane and the yz-plane in $\mathbb{R}^3$

Example. In $\mathbb{R}^3$, the subspace spanned by $(1,2,3)^\mathsf{T}$ is orthogonal to the subspace spanned by $(1,1,-1)^\mathsf{T}$.

Example. In $\mathbb{R}^3$, the subspace spanned by $(1,2,3)^\mathsf{T}$ is orthogonal to the subspace spanned by $\{(1,1,-1)^\mathsf{T}, (5,-4,1)^\mathsf{T}\}$.

Theorem 3.1.1 (3C). $A_{m\times n}$ The row space is orthogonal to the null space (in $\mathbb{R}^n$), and the column space is orthogonal to the left null space (in $\mathbb{R}^m$).

Proof. This is the proof.

1°
- $\mathbf{v} \in$ row space of A, then we have $\mathbf{b} = A^\mathsf{T} y$ for some $y \in \mathbb{R}^m$.
- $\mathbf{w} \in$ null space of A, then we have $A\mathbf{w} = 0$.

$$\mathbf{v}^\mathsf{T}\mathbf{w} = (A^\mathsf{T} y)^\mathsf{T}\mathbf{w} = y^\mathsf{T}(A\mathbf{w}) = y^\mathsf{T} 0 = 0$$

2°
- $b \in \mathcal{C}(A) \;\Rightarrow\; Ax = b$ is solvable.
- $y \in \mathcal{N}(A^\mathsf{T}) \Rightarrow A^\mathsf{T} y = 0$.

$$b^\mathsf{T} y = (Ax)^\mathsf{T} y = x^\mathsf{T}(A^\mathsf{T} y) = x^\mathsf{T} 0 = 0$$

■

Example.

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 5 & 2 & 7 \end{pmatrix}_{2\times 3} \longrightarrow U = \begin{pmatrix} \boxed{1} & 3 & 4 \\ 0 & \boxed{\text{-13}} & -13 \end{pmatrix}_{2\times 3}$$

- $\mathcal{C}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right\}, \quad \text{rank} = 2$
- $\mathcal{C}(A^\top) = \text{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 7 \end{pmatrix} \right\}, \quad \text{rank} = 2$
- $\mathcal{N}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}, \quad n - \text{rank} = 1$
- $\mathcal{N}(A^\top) = \text{span}\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}, \quad m - \text{rank} = 0$

$$\boxed{\mathcal{C}(A) \perp \mathcal{N}(A^\top) \in \mathbb{R}^2} \quad \boxed{\mathcal{C}(A^\top) \perp \mathcal{N}(A) \in \mathbb{R}^3}$$

Note. The nullspace $\mathcal{N}(A)$ doesn't contain "some" of vectors orthogonal to the row space. It contain "every" such vector.

Proposition 3.1.3. Let V be an inner product space, and let W be a subspace of V. Then the set is defined

$$U = \{\mathbf{v} \in V \mid \langle \mathbf{v}, \mathbf{w} \rangle = 0, \forall \mathbf{w} \in W\}$$

Then U is a subspace of V.

Definition 3.1.8. The subspace U is called the $\boxed{\text{orthogonal complement}}$ of W in V, denoted by $W^\perp$ (W-perp). By definition of nullspace $\mathcal{N}(A)$, we have

$$\mathcal{N}(A) = (\mathcal{C}(A^\top))^\perp, \quad \text{or} \quad \mathcal{C}(A) = (\mathcal{N}(A^\top))^\perp$$

Theorem 3.1.2 ((3D) Fundamental Theorem of Linear Algorithm)**.** The $\underbrace{\text{nullspace}}_{W}$ is the orthogonal complement of the $\underbrace{\text{row space}}_{V(W^\perp)}$ in $\underbrace{\mathbb{R}^n}_{V}$, and the left $\underbrace{\text{nullspace}}_{W}$ is the orthogonal complement of the $\underbrace{\text{column space}}_{V(W^\perp)}$ in $\underbrace{\mathbb{R}^m}_{V}$.

Proposition 3.1.4 (3E)**.** The equation $Ax = b$ is solvable if and only if

$$b^\top y = 0, \quad \forall y \in \mathcal{N}(A^\top)$$

Note. Solvability of $Ax = b$:

- Direct approach: b must be a combination of the columns of A.
- Indirect approach: b must be orthogonal to every vector that is orthogonal to the columns of A.

3.1.2 The Matrix and the Subspaces

U and W can be orthogonal without being complements when their dimensions are too small. In $\mathbb{R}^3$

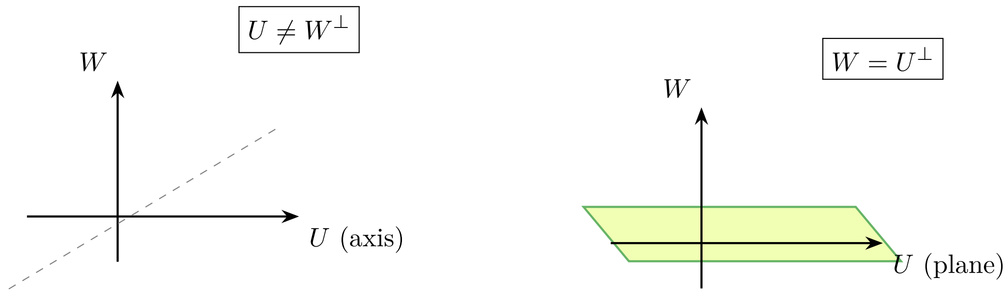


Figure 3.2: Orthogonal but not complements

$$W = U^{\perp} \;\Rightarrow\; U = W^{\perp} \text{ or } U^{\perp\perp} = U$$

When the space is split into orthogonal parts (i.e. $V = U+W = U+U^{\perp}$), so every vector ($x = x_U + x_{U^{\perp}}$).

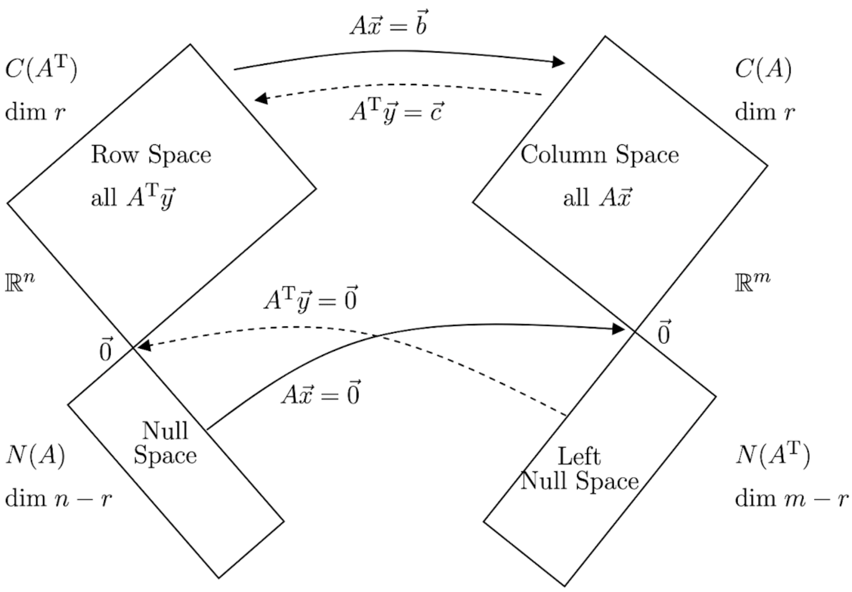


Figure 3.3: Fundamental Theorem of Linear Algebra

Proposition 3.1.5 (3F). The mapping from row space to column space is actually invertible. Every matrix $A_{m\times n}$ transforms its row space to its column space. (On these r-dimensional subspaces, A is invertible.)

$$A_{m\times n} : \underset{x}{\mathbb{R}^n} \xrightarrow{A} \underset{Ax=b}{\mathbb{R}^m} \qquad Ax = b$$

$$A^{\mathsf{T}}_{n\times m} : \underset{b}{\mathbb{R}^m} \xrightarrow{A^{\mathsf{T}}} \underset{A^{\mathsf{T}}b=0}{\mathbb{R}^n} \qquad A^{\mathsf{T}}b \overset{?}{=} x \qquad x = A^{-1}b$$

- $A^\top$ moves the space correctly but NOT the individual vectors.
- When A^{-1} fails to exist, we can substitute. It's called the pseudoinverse, denoted by A^+.

$$\begin{cases} A^+Ax = x, & \forall x \in \mathcal{C}(A^\top) \\ A^+b = 0, & \forall b \in \mathcal{N}(A^\top) \end{cases}$$

3.2 Inner Product and Projections onto Lines

Inner product $x^\top y = \begin{cases} = 0, & \text{if } x \perp y \\ \neq 0 \end{cases}$

Question. Practical applications?
Least squares solution to an overdetermined system. i.e. given a vector b not falling in the desired space, we have to project to the subspace. Then we get the approximate solution.

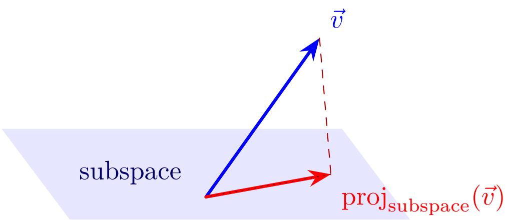
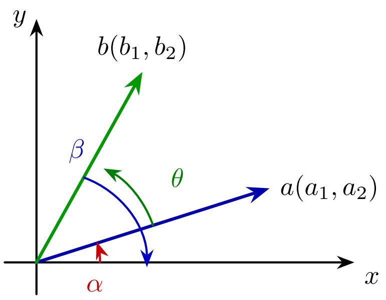


Figure 3.4: Projection onto a subspace (in $\mathbb{R}^3$)

Question. Practical applications?
A formula for the projection, we need the basis.

3.2.1 Inner Product and Schwarz Inequality



$$\sin\alpha = \frac{a_2}{\|\mathbf{a}\|}, \quad \cos\alpha = \frac{a_1}{\|\mathbf{a}\|},$$
$$\sin\beta = \frac{b_2}{\|\mathbf{b}\|}, \quad \cos\beta = \frac{b_1}{\|\mathbf{b}\|}.$$

Figure 3.5: Angle between two vectors in $\mathbb{R}^2$

$$\cos\theta = \cos(\beta - \alpha) = \cos\beta\cos\alpha + \sin\beta\sin\alpha = \frac{a_1b_1 + a_2b_2}{\|a\|\|b\|} = \frac{a^\top b}{\|a\|\|b\|}$$

Proposition 3.2.1 (3G). The cosine of the angle between any two vectors $a, b \in \mathbb{R}^n$ is

$$\cos\theta = \frac{a^\intercal b}{\|a\|\|b\|}$$

If we consider the relationship between $\|a\|, \|b\|$ and $\|b-a\|$, then we have

$$\|b-a\|^2 = \|a\|^2 + \|b\|^2 - 2\|a\|\|b\|\cos\theta \quad \text{(Law of Cosines)}$$

Projection onto a Line:

$$(b-p) \perp a \iff (b-p)^\intercal a = 0 \iff (b-\alpha a)^\intercal a = 0$$
$$\iff b^\intercal a - \alpha a^\intercal a = 0 \iff \alpha = \frac{a^\intercal b}{a^\intercal a}$$

Proposition 3.2.2 (3H). The projection of b onto the line through $0\&a$ is

$$p = \frac{a^\intercal b}{a^\intercal a} \cdot a$$

Theorem 3.2.1 (3I Schwarz Inequality). For any two vectors in inner product space satisfy the **Cauchy-Schwarz inequality**:

$$|a^\intercal b| \le \|\mathbf{a}\|\|\mathbf{b}\| \quad \text{ or } |\langle \mathbf{a}, \mathbf{b}\rangle| \le \|\mathbf{a}\|\|\mathbf{b}\|$$

with equality if and only if $b = \alpha a$, for some $\alpha \in \mathbb{F}$.

Proof.

$$\|b-p\|^2 = \|b - \frac{a^\intercal b}{a^\intercal a} a\|^2 = b^\intercal b - 2\cdot\frac{(a^\intercal b)^2}{a^\intercal a} + \left(\frac{a^\intercal b}{a^\intercal a}\right)^2 a a^\intercal$$
$$= \frac{(b^\intercal b)(a^\intercal a) - (a^\intercal b)^2}{a^\intercal a} \ge 0$$

$$\Rightarrow |a^\intercal b| \le \|a\|\|b\|$$

and the equality holds $\iff \|b-p\| = 0 \iff b = p = \alpha a$. ■

Example. Project $(1,1,1) \to (1,2,3)$

$$p = \frac{a^\intercal b}{a^\intercal a} a = \frac{6}{14}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}3/7\\6/7\\9/7\end{pmatrix}$$

3.2.2 Projections of Rank One

Question. What is the matrix this linear transformation that maps b to p

$$p = \frac{a^\intercal b}{a^\intercal a} \cdot a = \frac{a a^\intercal}{a^\intercal a} b$$

The projection matrix is $P = \dfrac{a a^\intercal}{a^\intercal a}$

Note. Here are some properties of P:

1° P is symmetric: $P^{\mathsf{T}} = P$

2° $P^2 = P$ (idempotent)

Proof. Here are the proofs:

1° $P^{\mathsf{T}} = \left(\frac{aa^{\mathsf{T}}}{a^{\mathsf{T}}a}\right)^{\mathsf{T}} = \frac{(a^{\mathsf{T}})^{\mathsf{T}}a^{\mathsf{T}}}{a^{\mathsf{T}}a} = \frac{aa^{\mathsf{T}}}{a^{\mathsf{T}}a} = P$

2° $P^2 = \left(\frac{aa^{\mathsf{T}}}{a^{\mathsf{T}}a}\right)\left(\frac{aa^{\mathsf{T}}}{a^{\mathsf{T}}a}\right) = \frac{aa^{\mathsf{T}}aa^{\mathsf{T}}}{(a^{\mathsf{T}}a)^2} = \frac{a(a^{\mathsf{T}}a)a^{\mathsf{T}}}{(a^{\mathsf{T}}a)^2} = \frac{aa^{\mathsf{T}}}{a^{\mathsf{T}}a} = P$

Proof complete. ■

- rank$(P) = 1$, nullspace of P is the space orthogonal to a. i.e.

$$\mathcal{N}(P) \perp \mathcal{C}(P)$$

which is not general. It is right here because $\mathcal{C}(P) = \mathcal{C}(P^{\mathsf{T}}) = \text{span}(a)$.

Remark (Scaling). Project b onto a, which can be scaled arbitrarily. i.e. project onto αa

$$p = \frac{a'a'^{\mathsf{T}}}{a'^{\mathsf{T}}a'} = \frac{(\alpha a)(\alpha a)^{\mathsf{T}}}{(\alpha a)^{\mathsf{T}}(\alpha a)} = \frac{\alpha^2 aa^{\mathsf{T}}}{\alpha^2 a^{\mathsf{T}}a} = \frac{aa^{\mathsf{T}}}{a^{\mathsf{T}}a} = p \quad \text{(remains the same)}$$

Lecture 10

3.3 Projections and Least Squares Applications

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$$\begin{cases} a_1 x = b_1 \\ a_2 x = b_2 \\ a_3 x = b_3 \end{cases} \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} x = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$E^2 = (a_1 x - b_1)^2 + (a_2 x - b_2)^2 + (a_3 x - b_3)^2$$

1° if $b = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} x_0$ then $E^2 = 0$

2° Or consider

$$\frac{dE^2}{dx} = 2[a_1(a_1 x - b_1) + a_2(a_2 x - b_2) + a_3(a_3 x - b_3)] = 0$$

$$\Rightarrow \bar{x} = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{a_1^2 + a_2^2 + a_3^2} = \underbrace{\boxed{\frac{a^T b}{a^T a}}}_{\alpha}$$

Hence, we call the projection of b onto a as $p_a b = \bar{x} a = \frac{a^T b}{a^T a} a$, which is also calles the least squares solution.

$$a^T(b - \bar{x}a) = 0 = a^T b - \bar{x} a^T a$$

3.3.1 Least Squares Problem with Several Variables

$$A_{m\times n}x_{n\times 1} = b_{m\times 1} \quad (m > n)$$

- If $b \in \mathrm{Col}(A)$, then the system is solvable.
- If the equations contain errors, then b might not belong to $\mathcal{C}(A)$

$$A_{3\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

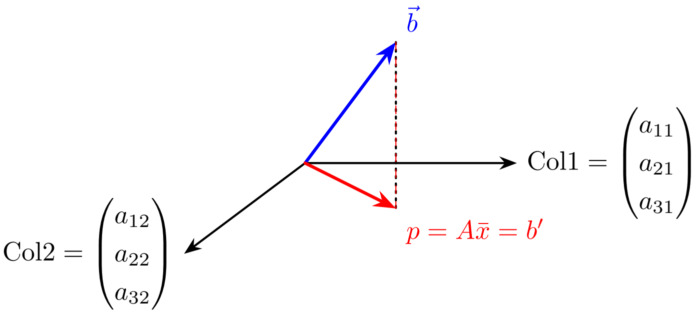

Figure 3.6: Projection of $\vec{b}$ onto the span of the columns of A

- $Ax = b$ has error, $b \notin \mathcal{C}(A)$
- $Ax = b'$ is solvable $\quad \Rightarrow b' \in \mathcal{C}(A) \;\Leftrightarrow\; \exists\ \bar{x}_{n\times 1} \ni A\bar{x} = p = b'$

Note. To find x, we do it in three ways:

1° The vectors perpendicular to $\mathcal{C}(A)$ are in $\mathcal{N}(A^T)$

$$A^T(b - A\bar{x}) = 0 \;\Rightarrow\; A^TA\bar{x} = A^Tb$$

2° The error vector must be perpendicular to each column of A.
If $A = [a_1\ a_2\ \cdots\ a_n]$

$$\begin{cases} a_1^T(b - A\bar{x}) = 0 \\ a_2^T(b - A\bar{x}) = 0 \\ \vdots \\ a_n^T(b - A\bar{x}) = 0 \end{cases} \quad \Rightarrow \quad A^T(b - A\bar{x}) = 0 \quad \Rightarrow \quad A^TA\bar{x} = A^Tb\ (A'x' = b')$$

3° The third way is to differentiate the sum of squares.

$$E^2 = \|Ax - b\|^2 = (Ax - b)^T(Ax - b) \quad \Rightarrow \quad A^TAx - A^Tb = 0 \quad \Rightarrow \quad A^TA\bar{x} = A^Tb$$

Proposition 3.3.1 (3L). The least-squares solution to an inconsistent system $Ax = b$ of m equations in n unknowns satisfies

$$\boxed{A^T A\bar{x} = A^T b}$$

The above equation is referred to the Normal Equation.

Note. The properties of $A^T A$:

1° $A^T A$ is symmetric.

Proof. $(A^T A)^T = A^T (A^T)^T = A^T A$ ∎

2° The $(i, j)^{\text{th}}$ entry of $A^T A$ is the inner product of the i^{th} and j^{th} columns of A.

3° $A^T A$ has the same nullspace of A (i.e., $\mathcal{N}(A^T A) = \mathcal{N}(A)$).

Proof. We follow the two directions:

- $Ax = 0 \quad \Rightarrow \quad A^T Ax = 0 \qquad \therefore\ \mathcal{N}(A) \subseteq \mathcal{N}(A^T A)$
- if $A^T Ax = 0$, then

$$x^T A^T Ax = (Ax)^T (Ax) = \|Ax\|^2 = 0 \quad \Rightarrow \quad Ax = 0 \qquad \therefore\ \mathcal{N}(A^T A) \subseteq \mathcal{N}(A)$$

Proof complete. ∎

4° $A^T A$ is positive definite, i.e., for any non-zero vector x,

$$x^T A^T Ax = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$$

with equality if and only if $Ax = 0$.

Proposition 3.3.2 (3L (conti.)). The least-squares solution to the inconsistent system $Ax = b$ is the solution of the normal equation

$$A^T A\bar{x} = A^T b$$

Proposition 3.3.3 (3M). If the columns of A are linearly independent (rank $= n$), then $A^T A$ is invertible and

$$\bar{x} = (A^T A)^{-1} A^T b$$

The projection of b onto $\mathcal{C}(A)$ is therefore

$$p_{\mathcal{C}(A)} = A\bar{x} = A(A^T A)^{-1} A^T b$$

Proof. We consider $\text{rank}(A) = r = n \ \Rightarrow\ \mathcal{N}(A) = \{0\} \ \Rightarrow\ \mathcal{C}(A^T A) = \{0\}$

$$\therefore\ \text{rank of } A^T A = n \text{ which means } A^T A \text{ has full rank.}$$

$$\therefore A^T A \text{ is invertible.}$$

∎

Note. If $\text{rank}(A) < n$, then $A^T A$ is singular and the linear system has infinitely many solutions.

Example.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{pmatrix}_{3\times 2}, \quad b = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}_{3\times 1}, \quad Ax = b$$

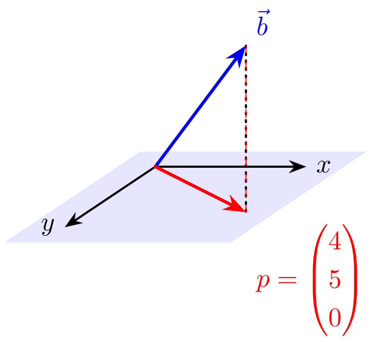

Figure 3.7: Projection of $\vec{b}$ onto the span of the columns of A

1° $\bar{x} = (A^T A)^{-1} A^T b \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow p = A\bar{x} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$

2° $\mathcal{C}(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\} = xy - \text{plane} \qquad \therefore p = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$

Remark. 1° The normal equation $A^T A\bar{x} = A^T b$ is indeed consistent.

2° If $b \in \mathcal{C}(A)$, then $p = b$.

3° Suppose $b \perp \mathcal{C}(A)$, then $p = 0$.

4° When A is square and invertiblem, then $\mathcal{C}(A) = \mathbb{R}^n$

$$p = A(A^T A)^{-1} A^T b = b$$

5° If $A_{m\times 1} = a$, then $A^T A = a^T a$

$$\bar{x} = (a^T a)^{-1} a^T b = \frac{a^T b}{a^T a} = \alpha$$

3.3.2 Projection Matrices

Let A be an $m \times n$ matrix over $\mathbb{R}$, $\mathcal{C}(A) \le \mathbb{R}^m$.
Let $b \notin \mathcal{C}(A)$, the closest point to b in $\mathcal{C}(A)$ is $p = A(A^T A)^{-1} A^T b$.
Let $\mathcal{P} = A(A^T A)^{-1} A^T$.
i.e. The matrix projects any vector b onto $\mathcal{C}(A)$.
i.e. $p = \mathcal{P}b$ is the component of b in $\mathcal{C}(A)$.
i.e. $b - \mathcal{P}b$ (error) is the component of b in orthogonal complement $\mathcal{N}(A^T)$.

Corollary 3.3.1.

$$I = \underbrace{\mathcal{P}}_{\text{projection onto } \mathcal{C}(A)} + \underbrace{(I - \mathcal{P})}_{\text{projection onto } \mathcal{C}(A)^{\perp}}$$

Theorem 3.3.1 (3N)**.** Here are some properties of projection matrix

1. $\mathcal{P}^2 = \mathcal{P}$
2. $\mathcal{P}^T = \mathcal{P}$

3.3.3 Least Square Fitting of Data

$$C + Dt = b$$

Given m data points

$$\begin{cases} C + Dt_1 = b_1 \\ C + Dt_2 = b_2 \\ \vdots \\ C + Dt_m = b_m \end{cases} \quad \Rightarrow \quad \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\begin{aligned} \Longrightarrow \min E^2 &= \|b - (C + Dt)\|^2 = \|b - Ax\|^2 \\ &= \sum_{i=1}^{m} (b_i - C - Dt_i)^2 \end{aligned}$$

Example. $(t_i, b_i) : (\underset{t_1}{-1}, \underset{b_1}{1}), (\underset{t_2}{1}, \underset{b_2}{1}), (\underset{t_3}{2}, \underset{b_3}{3})$

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad x = \begin{pmatrix} C \\ D \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad \Rightarrow \quad \bar{x} = \begin{pmatrix} \bar{C} \\ \bar{D} \end{pmatrix} = (A^T A)^{-1} A^T b = \begin{pmatrix} \frac{9}{7} \\ \frac{4}{7} \end{pmatrix}$$

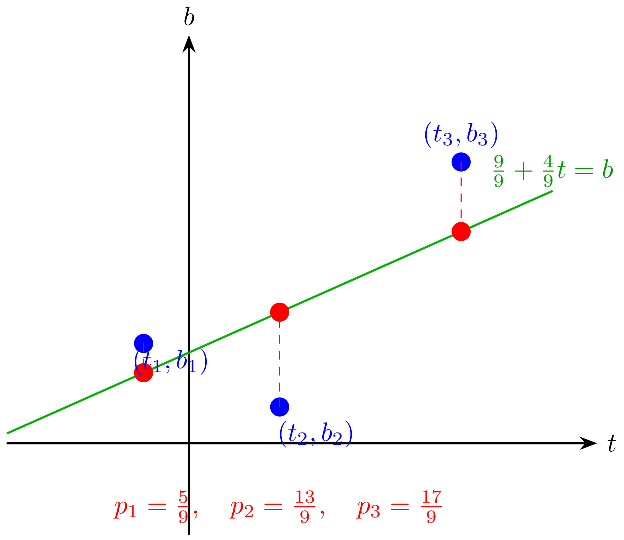

Figure 3.8: Least Squares Line Fitting

$$\mathcal{P} = A(A^TA)^{-1}A^T = \frac{1}{14}\begin{pmatrix} 13 & 3 & -2 \\ 3 & 5 & 6 \\ -2 & 6 & 10 \end{pmatrix}_{3\times 3} \quad \Rightarrow \quad p = \mathcal{P}b = \frac{1}{7}\begin{pmatrix} 5 \\ 13 \\ 17 \end{pmatrix}$$

$$\therefore \text{ error vector} = b - p = \frac{1}{7}\begin{pmatrix} 2 \\ -6 \\ 4 \end{pmatrix}$$

$$P = \frac{1}{14}\begin{pmatrix} 13 & 3 & -2 \\ 3 & 5 & 6 \\ -2 & 6 & 10 \end{pmatrix} \longrightarrow U = \frac{1}{14}\begin{pmatrix} \boxed{13} & 3 & -2 \\ 0 & \boxed{\frac{56}{13}} & \frac{84}{13} \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \mathcal{C}(P) = \text{span}\left\{\begin{pmatrix} 13 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}\right\} \Rightarrow x - 3y + 2z = 0$$

$$\mathcal{C}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}\right\} \Rightarrow x - 3y + 2z = 0 \quad \Rightarrow \quad \mathcal{C}(P) = \mathcal{C}(A)$$

3.4 Orthogonal Bases, Orthogonal Matrices and Gram-Schmidt Orthogonalization

Recall. The vectors $q_1, q_2, \cdots q_k$ are orthognormal if

$$q_i^T q_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

3.4.1 Orthogonal Matrices

Definition 3.4.1 (3Q). An orthognoral matrix Q is a square matrix satisfying $Q^TQ = I$. If $Q = [q_1\ q_2\ \cdots\ q_n]$, then

$$Q^TQ = \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{pmatrix}(q_1\ q_2\ \cdots\ q_n) = \begin{pmatrix} \underset{=1}{\boxed{q_1^Tq_1}} & \underset{=0}{\boxed{q_1^Tq_2}} & \cdots & q_1^Tq_n \\ q_2^Tq_1 & \underset{=1}{\boxed{q_2^Tq_2}} & \cdots & q_2^Tq_n \\ \vdots & \vdots & \ddots & \vdots \\ q_n^Tq_1 & q_n^Tq_2 & \cdots & q_n^Tq_n \end{pmatrix}_{n\times n} = I_n$$

i.e. The columns of Q are orthonormal and $Q^{-1} = Q^T$.

Example. Here are some examples:

- Rotation matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ YES
- Permutation matrix $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ YES

Proposition 3.4.1 (3R). Here are some properties

- $\|Qx\| = \|x\|, \quad \forall x$
- $\langle Qx, Qy \rangle = \langle x, y \rangle, \forall x, y$

The properties preserve

1° length

2° inner product

3° angle (since $\cos\theta = \dfrac{\langle x, y\rangle}{\|x\|\|y\|}$)

Remark. Since $Q^{-1} = Q^T$, we also have $QQ^T = I$. Therefore, the rows of a square matrix are orthonormal whenever the columns are orthonormal.

We've learned that any vector is a combination of basis vectors. The problem becomes how to find the coefficients of the combination.

Let $\{q_1, q_2, \cdots, q_n\}$ be an orthonormal basis, then for any vector b

$$b = x_1 q_1 + x_2 q_2 + \cdots + x_n q_n$$

try to compute x_i's:

$$q_1^T b = x_1 \underbrace{q_1^T q_1}_{1} + x_2 \underbrace{q_1^T q_2}_{0} + \cdots + x_n \underbrace{q_1^T q_n}_{0} = x_1$$

Similarly, we have $x_i = q_i^T b,\ i = 1, 2, \cdots, n.$ i.e.

$$\boxed{b = (q_1^T b) q_1 + (q_2^T b) q_2 + \cdots + (q_n^T b) q_n}$$

for the matrix form

$$b = (q_1\ q_2\ \cdots\ q_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = Qx \qquad x = Q^{-1} b = Q^T b = \begin{pmatrix} q_1^T b \\ q_2^T b \\ \vdots \\ q_n^T b \end{pmatrix}$$

Recall. $\mathcal{P} = \dfrac{a^T b}{a^T a} \cdot a$

Therefore, we can rewrite b as

$$\boxed{b = \frac{q_1^T b}{q_1^T q_1} q_1 + \frac{q_2^T b}{q_2^T q_2} q_2 + \cdots + \frac{q_n^T b}{q_n^T q_n} q_n = \mathcal{P}_{q_1} b + \mathcal{P}_{q_2} b + \cdots + \mathcal{P}_{q_n} b} \quad \text{(since } q_i^T q_i = 1\text{)}$$

i.e. The sum of the projections of b onto each basis vector equals to b itself.

Lecture 11

3.4.2 Rectangular Matrices with Orthogonal Columns

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$$Ax = b, \quad \text{where } A \text{ is not neccessarily square.}$$

Similarly, we may have a system $Qx = b$, where $Q_{m\times n}$ is NOT square and $m > n$.

Note.

$$Q^TQ = \begin{pmatrix} q_1^T \\ \vdots \\ q_n^T \end{pmatrix} \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} = I_n$$

In this case, Q^T is the left inverse of Q.

Proposition 3.4.2 (3S). If Q has orthonormal columns, then the least squares problem is easy.

- $Q_{m\times n}$: has no solution for most b $\quad\longleftrightarrow\quad Ax = b$
- $Q^TQ\bar{x} = Q^Tb$: normal equation $\quad\longleftrightarrow\quad A^TA\bar{x} = A^Tb$
- $\bar{x} = Q^Tb$: least squares solution
-

$$\begin{aligned} p = Q\bar{x} = QQ^Tb &= (q_1 \ \cdots \ q_n) \begin{pmatrix} q_1^T \\ \vdots \\ q_n^T \end{pmatrix} b \\ &= \sum_{i=1}^n (q_i^Tb)q_i : \text{ projection of } b \text{ onto } C(Q) \quad \longleftrightarrow \quad p = A\bar{x} \end{aligned}$$

- $P = QQ^T$ $\quad\longleftrightarrow\quad P = A(A^TA)^{-1}A^T$

3.4.3 The Gram-Schmidt Process

Recall. $S = \{x_1, \cdots, x_n\}$ is an orthognoral subset if V if $\forall i \neq j, \langle x_i, x_j\rangle = 0$ and S is orthonormal if additionally

$$\langle x_i, x_i\rangle = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Notation. $\|x\| = \sqrt{\langle x, x\rangle}$ is called the norm of x. ($\langle x, x\rangle > 0$, if $x \neq 0$)

Note. There are many norms,

- 1-norm: $\|x\|_1 = \sum_{i=1}^n |x_i|$
- 2-norm: $\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$
- ∞-norm: $\|x\|_\infty = \max_{1\leq i\leq n} |x_i|$

Theorem 3.4.1 (1). Let V be an inner product space and let $S = \{x_1 \cdots x_n\}$ be an orthogonal subset of non-zero vectors. If

$$y = \sum_{i=1}^{k} a_i x_i,$$

then

$$a_j = \frac{\langle y, x_j \rangle}{\|x_j\|^2} \quad \text{for } j = 1, \cdots, n. \qquad (\text{i.e. } y = \sum_{i=1}^{k} \frac{\langle y, x_j \rangle}{\|x_j\|^2} x_j)$$

Proof. Since

$$\langle y_i, x_j \rangle = \sum_{i=1}^{k} \quad = a_j$$

Thus,

$$a_j = \frac{\langle y, x_j \rangle}{\|x_j\|^2}.$$

■

Corollary 3.4.1 (1). If S is, then

$$y = \sum_{i=1}^{k} \langle y, x_j \rangle x_j.$$

Corollary 3.4.2 (2). If S is an orthonormal set of non-zero vectors, then S is linearly independent.

Example. In $\mathbb{R}^3$, $\left\{ \frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2) \right\}$. Find the orthognormal set.

Given $(1,2,3) = \frac{3}{\sqrt{2}} \left[\frac{1}{\sqrt{2}}(1,1,0) \right] + \frac{2}{\sqrt{3}} \left[\frac{4}{\sqrt{3}}(1,-1,1) \right] + \frac{2}{\sqrt{6}} \left[\frac{1}{\sqrt{6}}(-1,1,2) \right]$.

Remark. Suppose $\{y_1, y_2\}$ is linearly independent set. We would like to construct an orthogonal set, $\{x_1, x_2\}$, that spans the same subspace. One way is to take $x_1 = y_1$ and $x_2 = y_2 - p$, where p is the projection of y_2 onto y_1.

$$p = \frac{\langle y_2, y_1 \rangle}{\|y_1\|^2} y_1$$

In other words, we take

$$x_2 = y_2 - \frac{\langle y_2, y_1 \rangle}{\|y_1\|^2} y_1.$$

Theorem 3.4.2 (2. extend to n vectors). Let V be an inner product space and let $S = \{y_1, \cdots, y_m\}$ be a linearly independent subset of V. Define $S' = \{x_1, \cdots, x_m\}$ where

$$\boxed{x_1 = y_1, \quad x_k = y_k - \sum_{i=1}^{k-1} \frac{\langle y_k, x_i \rangle}{\|x_i\|^2} x_i, \quad \text{for } 2 \leq k \leq m.}$$

Gram-Schmidt Orthogonalization

THen S' is an orthogonal set of non-zero vectors such that

$$\text{span}(S') = \text{span}(S).$$

Proof. Supplymentary notes. ■

Example. In $\mathbb{R}^3$, let $y_1 = (1,1,0)$, $y_2 = (2,0,1)$, $y_3 = (2,2,1)$. Find an orthogonal basis for x_1, x_2, x_3.

- $x_1 = y_1 = (1,1,0)$
- $x_2 = y_2 - \frac{\langle y_2, x_1\rangle}{\|x_1\|^2} x_1 = (2,0,1) - \frac{2}{2}(1,1,0) = (1,-1,1)$
- $x_3 = y_3 - \frac{\langle y_3, x_1\rangle}{\|x_1\|^2} x_1 - \frac{\langle y_3, x_2\rangle}{\|x_2\|^2} x_2 = (-\frac{1}{3}, \frac{1}{3}, \frac{2}{3})$

3.4.4 The Factorization $A = QR$

$$A = (a_1 \mid \cdots \mid a_n)_{m\times n} \longrightarrow Q = (q_1 \mid \cdots \mid q_n)_{m\times n} \quad Q^T Q = I_n$$

Theorem 3.4.3 (3U). Every $m \times n$ matrix A with linearly independent columns can be factored as

$$A = Q_{m\times n} R_{n\times n}$$

The columns of Q are orthognormal and R is an invertible upper-triangular matrix. When $m = n$ and all matrices are square, Q is orthogonal.

Proof. We use Gram-Schmidt Orthogonalization process to construct Q and R.

> **As previously seen** (Theorem (2)).
>
> $$q_j' = a_j - \sum_{i=1}^{j-1} \frac{\langle a_j, q_i\rangle}{\|q_i\|^2} \cdot q_i, \quad q_j = \frac{q_j'}{\|q_j'\|}$$

Let $a_1, \cdots, a_n$ be the columns of A. By Gram-Schmidt Orthogonalization process, we can construct orthonormal vectors

$$q_1, \cdots, q_n \ni \text{span}\{q_1, \cdots, q_n\} = \text{span}\{a_1, \cdots, a_n\} \text{ for } j = 1, \cdots, n$$

So

$$a_j = (q^T a) \cdot q_1 + \cdots + (q_{j-1}^T a) \cdot q_{j-1} + \|q_j'\| \cdot q_j \quad \text{(i.e. linear combination of } q_j' s)$$

$$A = (a_1 \mid \cdots \mid a_n) = (q_1 \mid \cdots \mid q_n) \begin{pmatrix} \|q_1'\| & q_1^T a_2 & \cdots & q_1^T a_n \\ 0 & \|q_2'\| & \cdots & q_2^T a_n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \|q_n'\| \end{pmatrix} = QR$$

$$\begin{aligned} &j = 1: \ a_1 = \|q_1'\| \cdot q_1 \\ &j = 2: \ a_2 = (q_1^T a_2) \cdot q_1 + \|q_2'\| \cdot q_2 \\ &\vdots \\ &j = n: \ a_n = (q_1^T a_n) \cdot q_1 + (q_2^T a_n) \cdot q_2 + \cdots + (q_{n-1}^T a_n) \cdot q_{n-1} + \|q_n'\| \cdot q_n \end{aligned}$$

i.e. R is invertible since its diagonal entries are non-zero. ■

Example.

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -3 \\ 2 & 4 & 3 \end{pmatrix} \qquad a_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \; a_2 = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix}, \; a_3 = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix}$$

1° $q_1' = a_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \quad q_1 = \frac{q_1'}{\|q_1'\|} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

2° $q_2' = a_2 - \langle a_2, q_1 \rangle \cdot q_1 = \begin{pmatrix} -2 \\ 0 \\ 4 \end{pmatrix} - 2 \cdot \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -8 \\ -4 \\ 8 \end{pmatrix}, \quad q_2 = \frac{q_2'}{\|q_2'\|} = \frac{1}{12}\begin{pmatrix} -8 \\ -4 \\ 8 \end{pmatrix}$

3° $q_3' = a_3 - \langle a_3, q_1 \rangle \cdot q_1 - \langle a_3, q_2 \rangle \cdot q_2 = \begin{pmatrix} -3 \\ -3 \\ 3 \end{pmatrix} - (-1) \cdot \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} - 5 \cdot \frac{1}{3}\begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad q_3 = \frac{q_3'}{\|q_3'\|} = \frac{1}{1}\begin{pmatrix} 2/3 \\ -2/3 \\ 1/3 \end{pmatrix}$

4° Thus,

$$A = \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -3 \\ 2 & 4 & 3 \end{pmatrix} = \underbrace{\begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}}_{\text{orthognormal columns}} \underbrace{\begin{pmatrix} 3 & 2 & -1 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{invertible upper-triangular record Gram-Schmidt}} = QR$$

Remark. A: linearly independent columns

$$Ax = b \quad \text{inconsistent}$$

$$\begin{aligned} &\longrightarrow \quad A^T A\bar{x} = A^T b && (A = QR \to A^T A = R^T Q^T QR = R^T R) \\ &\longrightarrow \quad R^T R\bar{x} = A^T b && (R \text{ is invertible}) \\ &\longrightarrow \quad R\bar{x} = Q^T b && \end{aligned}$$

i.e. $\underset{Ax=b}{\text{inconsistent}} \to \underset{R\bar{x}=Q^Tb}{\text{consistent}}$

Chapter 4

Determinant

4.1 Introduction to Determinants

(A) A test for invertivility

$$\begin{cases} \text{If } \det A = 0, & A \text{ is singular} \\ \text{If } \det A \neq 0, & A \text{ is invertible} \end{cases}$$

The most important application is whether $\det(A - \lambda I) = 0$ (characteristic polynomial). We shall see that $\det(A - \lambda I)$ is a polynomial of degree n in λ.

(B) The determinant gives formulas for the pivots i.e.

$$\text{determinant} = \pm(\text{product of pivots})$$

(C) The determinant measures the dependence of $A^{-1}b$ on each entry of b (Cramer's rule). If one parameter in changed in an experiment, or one observation is corrected, the influence coefficients on $x = A^{-1}b$ is a ratio of determinants.

4.2 The Properties of Determinants

Definition 4.2.1 (determinant). Let A be an $n \times n$ square matrix over F. The determinant of A is a function

$$\det : M_{n \times n}(F) \to F$$

satisfies the following conditions:

(i) The $\det A$ is a linear function if the i-th row $(i = 1, 2, \cdots, n)$when the other $(n - 1)$ rows are held fixed. i.e. if

$$\det A = D(A_1, \cdots, A_i, \cdots, A_n) \text{where } A_i \text{ is the } i\text{-th row of } A,$$

then

$$\begin{aligned} &\det(A_1, \cdots, A_{i-1}, \alpha A_i + A_i', A_{i+1}, \cdots, A_n) \\ =&\alpha \det(A_1, \cdots, A_{i-1}, A_i, A_{i+1}, \cdots, A_n) + \det(A_1, \cdots, A_{i-1}, A_i', A_{i+1}, \cdots, A_n) \end{aligned}$$

Example.

$$\det\begin{pmatrix} a+a' & b+b' \\ c & d \end{pmatrix} = \det\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} + \det\begin{pmatrix} a & b' \\ c & d \end{pmatrix}$$

(ii) $\det I = 1$

(iii) $\det(P_{ij}A) = -\det A$, where $P_{ij}A$ is the permutation matrix.

(iv) $\det A = 0$, if A has two identical rows.

(v) $\det(EA) = \det A$, if E is the elementary operation of subtracting a multiple of one row from another row.

Proof. For the following steps,

$$\begin{aligned} &\det(A_1, \cdots, A_{i-1}, \alpha A_i + A_j, A_{i+1}, \cdots, A_n) \\ &\overset{\text{(i)}}{=} \alpha \det(A_1, \cdots, A_{i-1}, A_i, A_{i+1}, \cdots, A_n) + \det(A_1, \cdots, A_{i-1}, A_j, A_{i+1}, \cdots, A_n) \\ &\overset{\text{(iv)}}{=} \alpha \det(A) + 0 = \det(A) \end{aligned}$$

i.e. $\det(EA) = \det A$. ■

(vi) If A has a row of zeros, then $\det A = 0$.

Proof. (v) + (iv) ■

(vii) If A is triangular, then $\det A = a_{11}a_{22}\cdots a_{nn}$

Proof. Here is the steps

1° $\det A \overset{\text{(v)}}{=} \det\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \overset{\text{(i)}}{=} a_{11}\det\begin{pmatrix} a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix} = \cdots \overset{\text{(ii)}}{=} \prod_{i=1}^{n} a_{ii}$

2° If $a_{jj} = 0$, by (v), the j-th row can be converted to a zero row, thus by (vi), $\det A = 0$.

■

(viii) If A is singular $\Leftrightarrow$ $\det A = 0$. If A is invertible $\Leftrightarrow$ $\det A \neq 0$.

Proof. Let

$$A \xrightarrow{E_1E_2\cdots} U$$

$$\det A \overset{\text{(iii)}}{=} \det U \overset{\text{(vii)}}{=} \pm\, d_1 d_2 \cdots d_n$$

■

(ix) $\det(AB) = \det A \cdot \det B$

(x) $\det(A^T) = \det A$

Proof. We separately consider two cases:

- Case1: A is singular $\Leftrightarrow$ A^T is singular.
- Case2: A is nonsigular $\Rightarrow$ $PA = LDU$

  1° $(\det P)(\det A) = \det L \det D \det U = \det D$

  2° $(PA)^T = (LDU)^T$ and thus

$$(\det A^T)(\det P^T) = \det D^T \Rightarrow \det A^T = \det D = \det A$$

> **Note.** $PP^T = I \;\Rightarrow\; (\det P)(\det P^T) = \det I = 1$ and $\det P,\ \det P^T \in \{1, -1\}$

Done. ∎

Example.

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{n\times n} = L \begin{pmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{3}{2} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \frac{4}{3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n}{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{n+1}{n} \end{pmatrix} U$$

Thus,

$$\det A_n = 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n}{n-1} \cdot \frac{n+1}{n} = n + 1.$$

4.3 Formulas for the Determinant

Proposition 4.3.1 (4A)**.** If A is nonsingular, then $A = P^{-1}LDU$ and

$$\det A = \pm(\text{product of pivots})$$

Example.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \frac{ad-bc}{a} \end{pmatrix} \begin{pmatrix} 1 & b/a \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

Example.

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \\ &= \det \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \det \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \det \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \end{aligned}$$

Thus, the non-zero terms have to come in different columns

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$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \overset{(1,2,3)}{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{vmatrix}} + \overset{(2,3,1)}{\begin{vmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{vmatrix}} + \overset{(3,1,2)}{\begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{vmatrix}}$$

$$+ \overset{(2,1,3)}{\begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{vmatrix}} + \overset{(3,2,1)}{\begin{vmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{vmatrix}} + \overset{(1,3,2)}{\begin{vmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{vmatrix}}$$

$$= a_{11}a_{22}a_{33} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{12}a_{23}a_{31} \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} + a_{13}a_{21}a_{32} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$+ a_{12}a_{21}a_{33} \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + a_{13}a_{22}a_{31} \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + a_{11}a_{23}a_{32} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}$$

$\Rightarrow n!$ ways to permute the numbers $1, 2, \ldots, n$

Corollary 4.3.1.

$$\det(A) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)} \right)$$

where S_n is the set of all permutations on $\{1, 2, \ldots, n\}$ and $\operatorname{sgn}(\sigma)$ is the sign of the permutation σ.

$$|S_n| = n!$$

In other words $\det(A)$ is the sum of $n!$ terms and for each term, every row and column cintributes to exactly one element. So it is not difficult to see that

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

where

$$A_{1j} = (-1)^{1+j} M_{1j}$$

is the **cofactor** of a_{1j}, and M_{1j} is the submatrix of A obtained by deleting the 1-th row and j-th column. Similarly,

Proposition 4.3.2 (4B)**.**

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \ldots + a_{in}A_{in}$$

where

$$A_{ij} = (-1)^{i+j} M_{ij}$$

is the **cofactor** of a_{ij}. M_{ij} is the submatrix of A obtained by deleting the i-th row and j-th column.

Example.

$$A = \begin{pmatrix} 1 & 2 & 5 & 4 \\ 3 & 6 & 4 & 2 \\ 0 & 3_{32} & 0 & 4_{34} \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

$$\begin{aligned} \det A &= 3(-1)^{3+2} \cdot \det M_{32} + 4(-1)^{3+4} \cdot \det M_{34} \\ &= (-3) \begin{vmatrix} 1 & 5 & 4 \\ 3 & 4 & 2 \\ -1 & 2 & 3 \end{vmatrix} + (-4) \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ -1 & 2 & 2 \end{vmatrix} \\ &= (-3)[1(8) + 5(-1)(11) + 4(10)] + (-4)[1(-6) + 2(10)(-1) + 5(7)] \\ &= -15 \end{aligned}$$

$$\because \det A = \det A^T$$

so we can also expand along columns. i.e.

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \ldots + a_{nj}A_{nj}$$

4.4 Appplications of Determinants

(A) The computation of A^{-1}

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} A_{11} & A_{21} & \ldots & A_{n1} \\ A_{12} & A_{22} & \ldots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \ldots & A_{nn} \end{pmatrix}}_{\text{adj}(A) \text{ adjugate matrix}} = \det(A)I_n$$

$$a_{11}A_{21} + a_{12}A_{22} + \ldots + a_{1n}A_{2n} = \det(B)$$

$$B = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

Proposition 4.4.1 (4C).

$$A \cdot \text{adj}(A) = \det(A)I_n$$

If $\det(A) \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A)$$

If $\det(A) = 0$, then A is not invertible.

(B) The solution of system of linear equations

Theorem 4.4.1 (4D - Cramer's Rule). If A is an invertible $n \times n$ matrix, then the unique solution of the system of equations $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = A^{-1}\mathbf{b}$ and

$$x_j = \frac{\det(A_j)}{\det(A)} \quad \text{,where } B_j = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1,j-1} & b_1 & a_{1,j+1} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2,j-1} & b_2 & a_{2,j+1} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{n,j-1} & \underset{\text{j-th column}}{b_n} & a_{n,j+1} & \dots & a_{nn} \end{pmatrix}$$

Proof. Let

$$\det B_j = \sum_{i=1}^{n} b_i A_{ij}$$

Since A is invertible, by Proposition 4C, we have

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A)$$

Thus,

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} \det B_1 \\ \det B_2 \\ \vdots \\ \det B_n \end{pmatrix}$$

■

(C) Volume of parallelepipeds

$$AA^T = \begin{pmatrix} — \mathbf{a_1} — \\ — \mathbf{a_2} — \\ \vdots \\ — \mathbf{a_n} — \end{pmatrix} \begin{pmatrix} | & | & & | \\ \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_n} \\ | & | & & | \end{pmatrix} = \begin{pmatrix} \ell_1^2 & 0 & \dots & 0 \\ 0 & \ell_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \ell_n^2 \end{pmatrix} \quad \ell_i : \text{ length of } \mathbf{a_i}$$

$$\det(AA^T) = (\det A)^2 = \ell_1^2 \ell_2^2 \dots \ell_n^2$$

$$\therefore \text{ If rows of } A \text{ are mutually perpendicular, } |\det A| = \ell_1 \ell_2 \dots \ell_n$$

(D) A formula for pivots

Proposition 4.4.2 (4E). If A is factored into LDU, then upper left corners satisfy

$$A_k = L_k D_k U_k$$

For every k, the submatrix A_k is going through a Gaussian elimination of its own.

$$\begin{pmatrix} L_k & 0 \\ B & C \end{pmatrix} \begin{pmatrix} D_k & 0 \\ 0 & E \end{pmatrix} \begin{pmatrix} U_k & F \\ 0 & G \end{pmatrix} = \begin{pmatrix} L_k D_k U_k & L_k D_k F \\ B D_k U_k & B D_k F + CEG \end{pmatrix} = A$$

The pivot entries ate all nonzero whenever the numbers of $\det A_k$'s are all nonzero.

Note.

$$\det A_k = (\det L_k) \cdot (\det D_k) \cdot (\det U_k) = \det(D_k) = \det D_k = d_{11} d_{22} \dots d_{kk}$$

> **Notation.**
> $$d_k = \frac{\det A_k}{\det A_{k-1}} \quad \text{for } k = 1, 2, \ldots, n \quad (\det A_0 := 1)$$

Gaussian Elimination can be carried out without row exchanges if and only if leading submatrices $A_1, A_2, \ldots, A_n$ are all nonzero.

$$d_1 d_2 \ldots d_k = \frac{\det A_1}{\det A_0} \cdot \frac{\det A_2}{\det A_1} \cdot \ldots \cdot \frac{\det A_n}{\det A_{n-1}} = \det A_n = \det A$$

Chapter 5

Eigenvalues and Eigenvectors

5.1 Introduction

Question. What are the eigenvalues of a matrix and how useful are they?

Consider a matrix $A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$, then A can be treated as a linear transformation on $\mathbb{R}^2$ that maps each vector $\mathbf{v}$ to $T(\mathbf{v}) = \mathbf{u}$. i.e.

$$\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \overset{T}{\longmapsto} T(\mathbf{v}) = \mathbf{u} = A\mathbf{v}$$

and we can get

$$A\mathbf{v} = \lambda \mathbf{v}$$

Definition 5.1.1. Let A be an $n \times n$ matrix. If there exists a nonzero vector $\mathbf{v}$ s.t.

$$A\mathbf{v} = \lambda \mathbf{v}$$

for some scalar λ, then λ is called an **eigenvalue** of A and $\mathbf{v}$ is called an **eigenvector** of A corresponding to λ.

Theorem 5.1.1 (5A).

$$A\mathbf{v} = \lambda \mathbf{v} \Leftrightarrow \det(A - \lambda I) = 0$$

and for each eigenvalue λ exists at least one (nonzero) eigenvector $\mathbf{x}$ associated with it.

Proof. We separately prove the two directions.

$\Rightarrow$ By definition, $\exists$ nonzero vector $\mathbf{x}$ s.t. $A\mathbf{x} = \lambda \mathbf{x}$. This means,

$$A\mathbf{x} - \lambda I \mathbf{x} = 0$$

has nonzero solution, so $A - \lambda I$ must be singular. i.e.

$$\det(A - \lambda I) = 0$$

$\Leftarrow$ If $\det(A - \lambda I) = 0$, then $A - \lambda I$ has nontrivial solution(s) $\mathbf{v}$. Hence,

$$A\mathbf{v} = \lambda \mathbf{v}$$

implies that λ is an eigenvalue of A with eigenvector $\mathbf{v}$.

Proof complete. ■

Remark. Eigenvectors are (by definition) nonzero vectors and for each eigenvalue, its corresponding eigenvectors are **NEVER** unique. e.g.

$$A(\alpha\mathbf{v}) = \alpha A\mathbf{v} = \alpha\lambda\mathbf{v} = \lambda(\alpha\mathbf{v}) \quad \forall\alpha \neq 0$$

Note. An $n \times n$ matrix A can have at most n distinct (real or complex) eigenvalues.

Definition 5.1.2.

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A and the polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial** of A. For each eigenvalue λ, the **eigenspace** corresponding to λ is defined as

$$E_\lambda = \{\mathbf{v} \in \mathbb{R}^n : A\mathbf{v} = \lambda\mathbf{v}\}$$

which is the null space of $A - \lambda I$.

Example.

$$A = \begin{pmatrix} 1 & 1 \\ -2 & 4 \end{pmatrix}$$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{pmatrix} \Rightarrow |A - \lambda I| = (1-\lambda)(4-\lambda) - (-2)(1) = 0 \Rightarrow \lambda^2 - 5\lambda + 6 = 0$$

Example.

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\lambda = 3, 2$$

Example. Projection matrix

$$P = \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix}$$

$$\det(P - \lambda I) = \begin{vmatrix} 0.5-\lambda & 0.5 \\ 0.5 & 0.5-\lambda \end{vmatrix} = (\lambda - 1)\lambda = 0$$

1° $\lambda_1 = 1,\ P - \lambda_1 I = \begin{pmatrix} 0.5-1 & 0.5 \\ 0.5 & 0.5-1 \end{pmatrix} = \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix}$

$$\Rightarrow \begin{cases} \begin{pmatrix} -0.5 & 0.5 \\ 0.5 & -0.5 \end{pmatrix} x_1 = 0 \ \Rightarrow\ x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} x_2 = 0 \ \Rightarrow\ x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$$

2° $\lambda_2 = 0,\ P - \lambda_2 I = P$

Example. A is triangular

$$\det(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & 0 & \dots & 0 \\ 0 & a_{22}-\lambda & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn}-\lambda \end{vmatrix} = \prod_{i=1}^{n}(a_{ii}-\lambda) = 0$$

Therefore, the eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 5.1.2 (5B). The sum of the n eigenvalues equals the sum of the n diagonal entries:

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$$

Furthermore, the product of the n eigenvalues equals the product of the n diagonal entries:

$$\det(A) = \prod_{i=1}^{n} a_{ii} = \prod_{i=1}^{n} \lambda_i$$

Proof. We separately prove the two parts.

(1) $p_A(x) = (\lambda_1 - x)(\lambda_2 - x)\dots(\lambda_n - x) = (-x)^n + (\lambda_1 + \lambda_2 + \dots + \lambda_n)(-x)^{n-1} + \dots$
The coefficient of $(-x)^{n-1}$ in $p_A(x)$ is $\lambda_1 + \lambda_2 + \dots + \lambda_n$.

(2) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$$p_A(x) = \det(A - xI) = \det \begin{pmatrix} a_{11}-x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-x \end{pmatrix}$$

$= (a_{11} - x) \times C_{11} + a_{12} \times C_{12} + \dots + a_{1n} \times C_{1n},$ where C_{1j} is the cofactor of a_{1j}.

For $C_{1j}, \forall j = 2, 3, \dots, n$, the highest power of $(-x)$ is $n - 2$.

For example, $C_{12} = (-1)^{1+2} \det \begin{pmatrix} a_{21} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{33} - x & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n3} & \cdots & a_{nn} - x \end{pmatrix}$.

So $C_{1j}, \forall j = 2, 3, \ldots, n$ can't generate the $(-x)^{n-1}$ term.
The coefficient of $(-x)^{n-1}$ in $p_A(x)$ is equal to the coefficient of $(-x)^{n-1}$ in

$$(a_{11} - x) \times \det \begin{pmatrix} a_{22} - x & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} - x & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} - x \end{pmatrix}.$$

Similarly,

the coefficient of $(-x)^{n-1}$ in $(a_{11} - x) \times \det \begin{pmatrix} a_{22} - x & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} - x & \ddots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} - x \end{pmatrix}$

is equal to the coefficient of $(-x)^{n-1}$ in

$$(a_{11} - x)(a_{22} - x) \times \det \begin{pmatrix} a_{33} - x & a_{34} & \cdots & a_{3n} \\ a_{43} & a_{44} - x & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n3} & a_{n4} & \cdots & a_{nn} - x \end{pmatrix}.$$

Therefore, the coefficient of $(-x)^{n-1}$ in $p_A(x)$ will be equal to the coefficient of $(-x)^{n-1}$ in $(a_{11} - x)(a_{22} - x) \ldots (a_{nn} - x)$.
i.e. the coefficient of $(-x)^{n-1}$ in $p_A(x)$ is $a_{11} + a_{22} + \cdots + a_{nn} = tr(A)$.

By (1) (2), we have $\lambda_1 + \lambda_2 + \cdots + \lambda_n = tr(A)$.

Next, we prove the product part.

$$p_A(x) = \det(A - xI) = (\lambda_1 - x)(\lambda_2 - x) \ldots (\lambda_n - x).$$
$$\Rightarrow p_A(0) = \det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$$

■

Let us summarize some properties of eigenvalues and eigenvectors.

(1) To each eigenvalue, there is an eigenvector corresponding to it, and to each eigenvector, there is an eigenvalue corresponding to it.

(2) An eigenvalue can be zero. However, an eigenvector can never be the zero vector.

(3) If $Ax = \lambda x$, then $A(\alpha x) = \lambda(\alpha x)$

i.e. any scalar multiple of an eigenvector is still an eigenvector corresponding to the same eigenvalue. However, there can be independent eigenvectors associated with the same eigenvalue.

Theorem 5.1.3. The following statements are equivalent:

(a) λ is an eigenvalue of A.

(b) $\det(A - \lambda I) = 0$.

(c) $A - \lambda I$ is not singular.

(5) The eigenvalue of A are the roots of its characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.

(6) If λ is an eigenvalue of A, then the corresponding eigenvectors is the solution(s) of the linear system $(A - \lambda I)x = 0$.

(7) If λ is an eigenvalue of A, then the nullspace of $(A - \lambda I)$ is called the eigenspace corresponding to λ.

(8) λ may be a repeated root of the characteristic polynomial. Thus multiplicity of repetition is called the **algebraic multiplicity** of the eigenvalue. The dimension of the eigenspace corresponding to λ is called the **geometric multiplicity** of the eigenvalue.

(9) If A is a matrix over $\mathbb{R}$, A may have no eigenvalues in $\mathbb{R}$. e.g.

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

However, if we allow complex eigenvalues and eigenvectors, then every real matrix has at least one eigenvalue in $\mathbb{C}$.

Lecture 13

9 Dec. 13:20

5.2 Diagonalization of a Matrix

Definition 5.2.1. An $n \times n$ matrix A is said to be **diagonalizable** if there exists a nonsingular matrix S such that

$$S^{-1}AS = \Lambda$$

where Λ is a diagonal matrix.

Theorem 5.2.1 (5C). Suppose $A_{n\times n}$ has n linearly independent eigenvectors $x_1, x_2, \ldots, x_n$. Let S be the $n \times n$ matrix with $x_1, x_2, \ldots, x_n$ as its columns. Then

$$S^{-1}AS = \Lambda = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

where λ_i's satisfy $Ax_i = \lambda_i x_i$.

Proof. Suppose $S = (x_1 \mid \cdots \mid x_n)_{n\times n}$. Then

$$\begin{aligned} AS &= (Ax_1 \mid \cdots \mid Ax_n) = (\lambda_1 x_1 \mid \cdots \mid \lambda_n x_n)_{n\times n} \\ &= (x_1 \mid \cdots \mid x_n) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} = S\Lambda \\ &\Rightarrow S^{-1}AS = \Lambda \end{aligned}$$

for nonsingular S. ■

Remark (1). If $\lambda_1, \ldots, \lambda_n$ are distinct, then the eigenvectors $x_1, \ldots, x_n$ are linearly independent. In other words, a matrix with distinct eigenvalues can be diagonalized.

Remark (2). The diagnoalizing matrix S is not unique. Repeated eigenvalues leave more e.g.

$$S^{-1}IS = I$$

is true for any nonsingular S.

Remark (3). $AS = S\Lambda$ holds if and only if the columns of S are eigenvectors of A.

Remark (4). Note all matrices posses n linearly independent eigenvectors and therefore not all matrices are diagonalizable.

Theorem 5.2.2 (5D). The eigenvectors $x_1, \ldots, x_n$ coorsponding to the distinct eigenvalues $\lambda_1, \ldots, \lambda_k$ of A are linearly independent.

Example.

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

where $\lambda_1 = \lambda_2 = 1$

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ x_2 \end{pmatrix} \quad \Rightarrow \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ t \in \mathbb{R}$$

Let $P = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$, when we have

$$AP = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = PD$$

which implies that A is not diagonalizable.

Note. We have following properties

1° Diagonalizability is connected to eigenvectors (n linearly independent eigenvectors).

2° Invertibility is connected to eigenvalues (no zero eigenvalue).

The only connection between diagonalizability and invertibility probably is

"Diagonalization can fail only if there are repeated eigenvalues."

Example.

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$$

We have

$$\begin{cases} A^T = A \\ A^2 = A \end{cases} \Rightarrow A \text{ is a projection matrix where the eigenvalues are } 0, 1$$

- $\lambda = 1$, we have eigenvector $x_1 = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $\lambda = 0$, we have eigenvector $x_2 = t \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \Rightarrow \quad S^{-1}AS = \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Example.

$$Q = I - 2uu^T, \quad u \in \mathbb{R}^n, \quad u^T u = 1$$

is called a **Householder reflection matrix**. (Reflection about the hyperplane orthogonal to u direction.)

Assume

$$u = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \Rightarrow Q = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \qquad Q\bar{x} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

We have

$$\det(Q - \lambda I) = \lambda^2 - 1 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = -1$$

- $\lambda_1 = 1$, eigenvector $x_1 = t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$
- $\lambda_2 = -1$, eigenvector $x_2 = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

The Householder transformation is a reflection about the axis perpendicular to u.

5.2.1 Powers and Products: A^k and AB

If $Ax = \lambda x$, $x \neq 0$

$$\Rightarrow A^2x = A(Ax) = \lambda Ax = \lambda^2 x$$

Proposition 5.2.1 (5E). The eigenvalues of A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ i.e. the k-th power of the eigenvalues of A.

- If $S^{-1}AS = \Lambda$, then $S^{-1}A^kS = \Lambda^k$.
- If A is invertible, then the eigenvalues of A^{-1} are $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$ and $S^{-1}A^{-1}S = \Lambda^{-1}$

Note. λ is an eigenvalue of A and μ is an eigenvalue of B and x is an eigenvector of B.

$$(AB)x = \mu Ax = \mu\lambda x = (\lambda\mu)x$$

but in general x is not necessarily an eigenvector of A corresponding to λ.

Example.

$$AB = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which has eigenvalues 1 and 0, but neither A nor B has any eigenvalues 1.

Note.

$$\begin{cases} \lambda : \text{eigenvalue of } A \\ \mu : \text{eigenvalue of } B \end{cases} \quad \begin{cases} \Rightarrow \lambda\mu : \text{may not be an eigenvalue of } AB \\ \Rightarrow \lambda + \mu : \text{may not be an eigenvalue of } A + B \end{cases}$$

Theorem 5.2.3 (5F). If A and B are diagonalizable , they have the same eigenvector matrix S if and only if they commute i.e. $AB = BA$.

Proof. We follow the two directions.

"$\Rightarrow$" If $\exists\ S \ni S^{-1}AS = \Lambda_1, S^{-1}BS = \Lambda_2$, then we have

$$AB = S\Lambda_1S^{-1}S\Lambda_2S^{-1} = S\Lambda_2\Lambda_1S^{-1} = S\Lambda_2S^{-1}S\Lambda_1S^{-1} = BA$$

"$\Leftarrow$" We assume that all eigenvalues of A are distinct. If $AB = BA$, and $Ax = \lambda x$, then

Case 1 $Bx = 0$, i.e. x is an eigenvector of B corresponding to eigenvalue 0.

Case 2 $Bx \neq 0$, then

$$ABx = BAx = \lambda Bx \Rightarrow Ax' = \lambda x'$$

So,

$$x' = Bx = \mu x \text{ i.e. } x \text{ is an eigenvector of } B.$$

Hence, A, B share the same eigenvectors.

Proof is complete. ■

Theorem 5.2.4. Let A be an $n \times n$ matrix over F. Assume. that the characteristic polynomial of A has solutions in F. Then for each eigenvalue λ of A, its geometric multiplicity is less than or equal to its algebraic multiplicity.

Theorem 5.2.5. AB and BA have the same eigenvalues.

5.3 Difference Equations and Powers A^k

- Difference equations: move forward in a finite # of finite steps.
- Differential equations: take infinite # of infinitesimal steps.

Example. Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, \cdots$$

$$\begin{cases} F_0 = 0 \\ F_1 = 1 \\ F_{k+2} = F_{k+1} + F_k, \ k \geq 0 \end{cases}$$

What is $F_{10000000000}$?

Let $u_k = \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$, $u_{k+1} = \begin{pmatrix} F_{k+2} \\ F_{k+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{k+1} \\ F_k \end{pmatrix}$

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, then we have

$$\boxed{u_{k+1} = Au_k} \Rightarrow u_k = A^k u_0$$

where $u_0 = \begin{pmatrix} F_1 \\ F_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Proposition 5.3.1 (5G). If A can be diagonalized, say $A = S\Lambda S^{-1}$, then

$$u_k = A^k u_0 = S\Lambda^k S^{-1} u_0 = S\Lambda^k C$$

where $C = S^{-1}u_0$ is a constant vector. Then,

$$u_k = S\Lambda^k C = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \boxed{\sum_{i=1}^{n} c_i \lambda_i^k x_i}$$

i.e. the solution is a linear combination of $\lambda_i^k x_i$.

Proposition 5.3.2 (5H). If $u_0 = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ where x_i's are eigenvectors of A corresponding to eigenvalues λ_i's, then

$$u_k = A^k u_0 = c_1\lambda_1^k x_1 + c_2\lambda_2^k x_2 + \cdots + c_n\lambda_n^k x_n$$

In general, u_0 is not an eigenvector but if u_0 is a linear combination of eigenvectors, then u_k is the same linear combination of $\lambda_i^k x_i$.

Note. To solve the difference equation $u_{k+1} = Au_k$, u_0 is given.

1° Find λ_i's and x_i's of A.

2° Let $S = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix}$, find $C = S^{-1}u_0$.

3° The solution is

$$u_k = S\Lambda^k C = \sum_{i=1}^{n} c_i\lambda_i^k x_i$$

5.3.1 Markov Process

Suppose each year 1/10 of the population moves in California and 2/10 moves out of California to other states. Let y be the peeple outside California and z be the people inside California, then at the end of the of the 1st year, we have

$$\begin{cases} y_1 = \frac{9}{10}y_0 + \frac{2}{10}z_0 \\ z_1 = \frac{1}{10}y_0 + \frac{8}{10}z_0 \end{cases} \Rightarrow \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

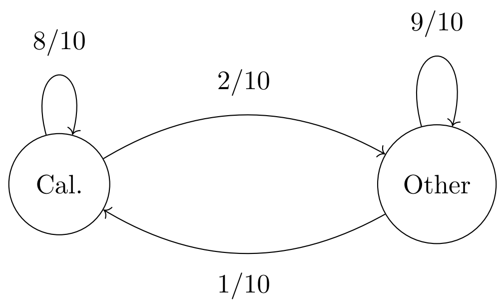

Figure 5.1: Markov Process

The essential assumption of Markov process is

- The population in both states is constant and never be negative.
- The u_{k+1} only depends on u_k i.e.

$$u_{k+1} = Au_k$$

- The total population is constant.
  1. all entries are positive or zero.
  2. column sums are 1.

Example.

$$A = \begin{pmatrix} 9/10 & 2/10 \\ 1/10 & 8/10 \end{pmatrix}$$

We have

$$\det(A - \lambda I) = \lambda^2 - \frac{17}{10}\lambda + 1 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = \frac{7}{10}$$

Since

$$A = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 7/10 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \quad (A = S\Lambda S^{-1})$$

We have

$$\begin{pmatrix} y_k \\ z_k \end{pmatrix} = A^k \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \begin{pmatrix} 1^k & 0 \\ 0 & (7/10)^k \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} y_0 \\ z_0 \end{pmatrix} \qquad c = \begin{pmatrix} y_0 + z_0 \\ y_0 - 2z_0 \end{pmatrix}$$

$$= (y_0 + z_0)(1)^k \cdot \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} + (y_0 - 2z_0)(7/10)^k \cdot \begin{pmatrix} 1/3 \\ -1/3 \end{pmatrix}$$

When $k \to \infty$, we have

$$\lim_{k\to\infty} \begin{pmatrix} y_k \\ z_k \end{pmatrix} = (y_0 + z_0) \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$$

i.e. No matter what the initial population distribution is, the population will eventually stabilize at 2/3 outside and 1/3 inside.

$$\begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix} \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} \quad \text{or} \quad Au_\infty = u_\infty$$

The **steady state** u_∞ is an eigenvector of A corresponding to $\lambda = 1$.

Appendix

Appendix A

SVD and Applications

A.1 Singular Value Decomposition (SVD)

Definition A.1.1 (Singular Value Decomposition)**.** Any matrix $A \in \mathbb{R}^{m \times n}$ can be factored into

$$A = U\Sigma V^T = (\text{orthogonal})(\text{diagonal})(\text{orthogonal})$$

where:

- The columns of $U \in \mathbb{R}^{m \times m}$ are eigenvectors of AA^T and satisfy $U^T U = I$.
- The columns of $V \in \mathbb{R}^{n \times n}$ are eigenvectors of $A^T A$ and satisfy $V^T V = I$.
- When A has rank r, the diagonal matrix $\Sigma \in \mathbb{R}^{m \times n}$ has r singular values, $\sigma_1 \geq \cdots \geq \sigma_r > 0$, filling the first r places on the main diagonal. The rest of Σ is zero.

Remark. The r singular values are the square roots of the nonzero eigenvalues of both AA^T and $A^T A$.

Theorem A.1.1 (Fundamental Subspaces Basis)**.** U and V give orthonormal bases for all four fundamental subspaces:

- First r columns of U: Basis for the column space of A.
- Last $m - r$ columns of U: Basis for the left nullspace of A.
- First r columns of V: Basis for the row space of A.
- Last $n - r$ columns of V: Basis for the nullspace of A.

Remark (1)**.**

$$AA^T = U\Sigma V^T V \Sigma^T U^T = U(\Sigma\Sigma^T)U^T$$

Here, $\Sigma\Sigma^T$ is the $m \times m$ eigenvalue matrix with $\sigma_1^2, \ldots, \sigma_r^2$ on the diagonal.

Remark (2)**.**

$$A^T A = V\Sigma^T U^T U \Sigma V^T = V(\Sigma^T\Sigma)V^T$$

Here, $\Sigma^T\Sigma$ is the $n \times n$ eigenvalue matrix with $\sigma_1^2, \ldots, \sigma_r^2$ on the diagonal.

Remark (3)**.** We can express A as a sum of rank-1 matrices:

$$A = \sum_{j=1}^{r} \sigma_j u_j v_j^T$$

where u_j is the j-th column of U and v_j is the j-th column of V.

Remark (4)**.** The action of A on v_j is given by $Av_j = \sigma_j u_j$.

Theorem A.1.2 (Procedure to Find SVD)**.** To find the SVD of a matrix A:

1. Calculate $A^T A$.
2. Find the eigenvalues of $A^T A$: $\sigma_1^2 \geq \cdots \geq \sigma_r^2 > 0 = \sigma_{r+1}^2 = \cdots = \sigma_n^2$.
3. Construct Σ by placing $\sigma_1, \ldots, \sigma_r$ on the diagonal and zeros elsewhere.
4. Find the eigenvectors for $A^T A$. For eigenvectors with the same eigenvalue, use Gram-Schmidt orthogonalization.
5. Construct $V = [v_1 \ldots v_n]$ where v_j is the normalized eigenvector corresponding to σ_j^2.
6. Construct $U = [u_1 \ldots u_m]$:
   - For $1 \leq j \leq r$, calculate $u_j = \frac{1}{\sigma_j} A v_j$.
   - For the remaining columns $(u_{r+1}, \ldots, u_m)$, find an orthonormal basis for the nullspace of A^T (Left Nullspace) using Gram-Schmidt.

Example. Find the SVD for $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \end{pmatrix}$.

1. Calculate $A^T A = \begin{pmatrix} 2 & -2 & 4 \\ -2 & 2 & -4 \\ 4 & -4 & 8 \end{pmatrix}$.
2. Eigenvalues of $A^T A$: $\sigma_1^2 = 12$, $\sigma_2^2 = 0$, $\sigma_3^2 = 0$.
3. $\Sigma = \begin{pmatrix} \sqrt{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.
4. Eigenvectors:
   - For $\lambda = 12$: $N(A^T A - 12I) = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \right\}$.
   - For $\lambda = 0$: $N(A^T A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}$. After Gram-Schmidt: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$.
5. Construct V:

$$v_1 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

Thus, $V = \begin{pmatrix} \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{-2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$.

6. Construct U:
   - $u_1 = \frac{1}{\sqrt{12}} A v_1 = \begin{pmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.
   - Find u_2 from $N(A^T) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$. Normalized: $u_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

   Thus, $U = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$.

A.2 Applications of SVD

A.2.1 Image Processing

An image can be represented as an $m \times n$ matrix of pixels. We can use SVD to find the essential information and compress the image.

- Typically, some singular values σ are significant while others are extremely small.
- We can keep the first k largest singular values and discard the rest. The approximation is:

$$A \approx \sum_{i=1}^{k} \sigma_i u_i v_i^T$$

- This reduces the data from $m \times n$ to $k(m + n + 1)$, saving storage/bandwidth.

A.2.2 Information Retrieval (Latent Semantic Indexing)

Definition A.2.1 (Term-by-Document Matrix)**.** Construct a matrix $A = [a_{i,j}]$ where $a_{i,j}$ represents the frequency of term i in document j.

Example (Search Engine Query)**.** Consider a matrix A representing terms (Advisor, Algebra, Ball, Calculus, Computer, Math) across 4 documents.
A query for "Club" can be processed by projecting terms and documents into a lower-dimensional space using SVD ($k = 2$).
The projection of terms is given by $U_k \Sigma_k$, and the projection of documents is given by $V_k \Sigma_k$.

- **Result:** The projection of the term "Club" and "Doc3" are found to be close in the 2D space, indicating relevance even if the exact word counts are sparse.